MATHEMATICS 3000

Science
OPTION

Québec Education Program

SOLUTIONS
Secondary 5

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Legal deposit

ISBN 978-2-7601-7108-4

Bibliothèque et Archives nationales du Québec, 2009
Bibliothèque et Archives Canada, 2009

PRINTED IN CANADA

We acknowledge the financial support of the Government of Canada through the Book Publishing Industry Development Program for our publishing activities.

Canada

Introduction

At the beginning of the third millennium, Guérin, éditeur is pleased to make available to Quebec teachers the exercise book, Math 506, **Science option**, of the **Mathematics 3000** collection.

This is an exercise book whose content, in accordance with the Quebec Education Program, is geared towards skill development, especially the three disciplinary skills: "**Solving a situational problem**", "**Using mathematical reasoning**", and "**Communicating by using mathematical language**".

Each book in the collection is divided into chapters that cover the various fields of mathematics such as arithmetic, algebra, geometry, probabilities and statistics.

Each chapter begins with the **Challenge** section where the student is invited, alone or in a team, to **solve situational problems** that have not been presented previously. The solution of each situation requires a combination of rules or principles that the student may have learned or not. In this section, the student is confronted with various situations that will provide him with the motivation to seek inside the chapter the elements allowing him to solve them.

Each of the other sections of a chapter starts with **learning activities** where the student is led step by step to the discovery of the concepts. Activities lead to highlighted **sections** summarizing the essential material of the course, and supported by **examples**. The student will find, in these highlights, complete references that will be useful throughout his learning process. The highlights are followed by a series of graded **exercises and problems** that will allow the student to **develop his skills** by solving situational problems, by using mathematical reasoning and by communicating using mathematical language. Each time the situation allows it, the student will have to explain the steps he used, justify his reasoning and finally communicate his answer in an appropriate manner.

Each chapter ends with an **Evaluation** section that will allow the student to ascertain if the knowledge has been acquired and if the skills have been attained.

A detailed **list of symbols** and an **index** at the end of each book will allow the student to easily find everything he needs during his learning.

This pedagogical tool, focused on skill development, is written in a clear and simple language and aims to be accessible to every student without sacrificing mathematical rigor.

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Chapter 1

Arithmetic and algebraic expressions

1.1 Powers with integer exponent

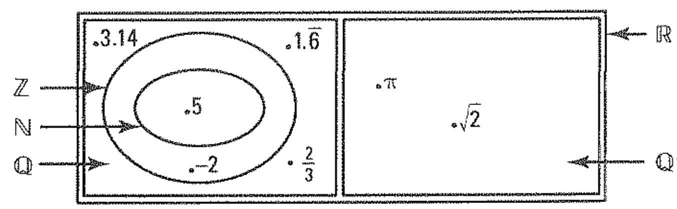
ACTIVITY 1 Real numbers

a) Complete the following with the term “rational” or “irrational”.

1. Any real number that has an infinite repeating decimal representation is ***rational.***
2. Any real number that has an infinite non-repeating decimal representation is ***irrational.***

b) Place the following real numbers in the Venn diagram below.

$-2; 5; \pi; \frac{2}{3}; 3.14; \sqrt{2}; 1.\overline{6}$



c) We represent the set of rational numbers by $\mathbb{Q}$ and the set of irrational numbers by $\mathbb{Q}'$.

Complete. 1. $\mathbb{Q} \cap \mathbb{Q}' =$ $\emptyset$ 2. $\mathbb{Q} \cup \mathbb{Q}' =$ $\mathbb{R}$

d) 1. If a is a natural number, what condition must be set on the number a such that $\sqrt{a}$ is a rational number? ***a must be a perfect square number.***

2. Complete the following using the appropriate symbol $\mathbb{Q}$ or $\mathbb{Q}'$.

1) $\sqrt{16} \in$ $\mathbb{Q}$ 2) $\sqrt{17} \in$ $\mathbb{Q}'$

REAL NUMBERS

- Any real number with an **infinite repeating** decimal representation is **rational.** Any real number with an **infinite non-repeating** decimal representation is **irrational.**
- $\mathbb{Q}$ represents the set of **rational** numbers, $\mathbb{Q}'$ the set of **irrationals** and $\mathbb{R}$ the set of **real numbers.** We have:

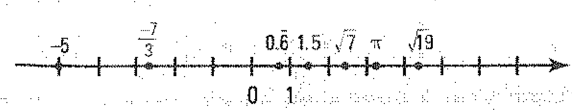
$\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$ $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$

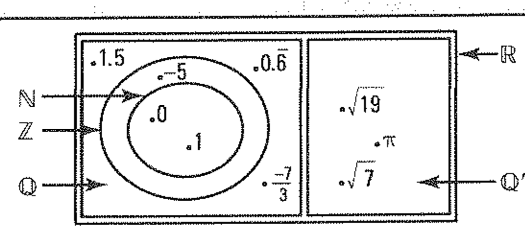
$\mathbb{R}_+$ represents the set of **positive** real numbers including zero.

$\mathbb{R}_-$ represents the set of **negative** real numbers including zero.

$\mathbb{R}^*$ represents the set of **non-zero** real numbers.

- Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.





$\mathbb{N}$ represents the set of natural numbers.
$\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{Z}$ represents the set of integers.
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
We have: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

ACTIVITY 2 Powers with integer exponent

a) Consider the power a^n. Complete.

1. a is called: *base*
2. n is called: *exponent*

b) 1. What is the definition of a^n? ($n \in \mathbb{N}, n \geq 2$) $a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$

2. Complete. 1) $a^1 =$ *a* 2) $a^0 =$ *1* ($a \neq 0$)

c) Calculate.

1. 2^4 *16*
2. $(-2)^4$ *16*
3. 2^3 *8*
4. $(-2)^3$ *-8*

d) Under what conditions will the power a^n be negative? *$a < 0$ and n is odd*

e) Complete.

1. $a^m \times a^n =$ a^{m+n}
2. $a^m \div a^n =$ a^{m-n}
3. $(a \times b)^n =$ a^nb^n
4. $\left(\frac{a}{b}\right)^n =$ $\frac{a^n}{b^n}$
5. $(a^m)^n =$ a^{mn}

f) What is the definition of a^{-n} when a is non-zero? $a^{-n} = \frac{1}{a^n}$

g) Calculate.

1. 2^{-2} $\frac{1}{4}$
2. $(-2)^{-2}$ $\frac{1}{4}$
3. 2^{-3} $\frac{1}{8}$
4. $(-2)^{-3}$ $-\frac{1}{8}$

POWERS WITH INTEGER EXPONENT

- **Definitions**

$a^n = \underbrace{a \times a \times \ldots \times a}_{\text{n factors}}$ ($n \in \mathbb{N}$ and $n \geq 2$) | $a^1 = a$ | $a^0 = 1$ ($a \neq 0$) | $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$)

- **Laws of exponents**

1. Product of two powers with the same base: $a^m \times a^n = a^{m+n}$
2. Quotient of two powers with the same base: $\frac{a^m}{a^n} = a^{m-n}$
3. Power of a product: $(ab)^n = a^nb^n$
4. Power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
5. Power of a power: $(a^m)^n = a^{mn}$

1 Calculate the following.

a) $(-2)^5$ *-32*
b) 2^5 *32*
c) $(-3)^4$ *81*
d) -3^4 *-81*
e) 3^{-2} $\frac{1}{9}$
f) $(-3)^{-2}$ $\frac{1}{9}$
g) $(-5)^{-2}$ $\frac{1}{25}$
h) -5^{-2} $-\frac{1}{25}$
i) $\left(-\frac{2}{3}\right)^2$ ~~$\frac{9}{4}$~~ $\frac{4}{9}$
j) $\left(-\frac{2}{3}\right)^{-2}$ $\frac{9}{4}$
k) $-\left(-\frac{3}{4}\right)^{-2}$ $-\frac{16}{9}$
l) $-\left(-\frac{3}{4}\right)^{-1}$ $\frac{4}{3}$

2. Using the laws of exponents, write the following as a single power and then evaluate it.

a) $(-3)^2 \times (-3)^3 = (-3)^5 = -243$ b) $2^3 \times 2^{-5} \times 2 = 2^{-1} = \frac{1}{2}$ c) $\left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^{-2} = \left(\frac{3}{4}\right)^0 = 1$

d) $\frac{(-5)^5}{(-5)^3} = (-5)^2 = 25$ e) $\frac{2^3}{2^5} = 2^{-2} = \frac{1}{4}$ f) $(2^3)^2 = 2^6 = 64$

g) $[(-2)^2]^4 = (-2)^8 = 256$ h) $(-2^3)^2 = 2^6 = 64$ i) $-(2^3)^2 = -2^6 = -64$

3. Simplify the following expressions using the laws of exponents.

a) $a^2 \times a^3 = a^5$ b) $(a^3)^2 = a^6$ c) $\frac{a^5}{a^2} = a^3$

d) $\left(\frac{a^3}{b^2}\right)^2 = \frac{a^6}{b^4}$ e) $(2a^3b^2)^2 = 4a^6b^4$ f) $(3a^2b) \cdot (2a^3b^2) = 6a^5b^3$

4. Simplify the following expressions using the laws of exponents.

a) $(3x^2)^2 = 9x^4$ b) $(-2x^3y^2)^2 = 4x^6y^4$ c) $\left(\frac{3x}{2y}\right)^3 = \frac{27x^3}{8y^3}$

d) $\left(\frac{2x^3}{3y^2}\right)^2 = \frac{4x^6}{9y^4}$ e) $\frac{12x^3y^2}{6x^2y^4} = \frac{2x}{y^2}$ f) $\frac{(2x^2y)^3}{4x^4y^2} = 2x^2y$

5. Simplify the following expressions using the laws of exponents.

a) $(2a)^3(3a)^2 = 72a^5$ b) $(2a^2b)^3 \cdot (3ab^2)^2 = 72a^8b^7$ c) $\frac{12a^3b^2}{6a^2b^4} = \frac{2a}{b^2}$

d) $\left(\frac{-2a^2}{3b}\right)^2 = \frac{4a^4}{9b^2}$ e) $\left(\frac{2a^2}{3b}\right)^3 \cdot \left(\frac{3a^2}{2b^2}\right)^3 = \frac{a^{9}}{b^9}$ f) $\left(\frac{2a^2}{3b}\right)^3 \div \left(\frac{a^2}{b}\right)^3 = \frac{8}{27}$

6. Simplify the following expressions using the laws of exponents.

a) $2a^{-2} \times 3a = \frac{6}{a}$ b) $(2a^{-2})^{-3} = \frac{a^6}{8}$ c) $\frac{12a^{-2}}{4a^{-5}} = 3a^3$

d) $(2a^{-3})^{-1} \times (3^{-1}a^2)^{-2} = \frac{9}{2a}$ e) $\left(\frac{2a^{-1}}{3a^2}\right)^{-3} = \frac{27a^9}{8}$ f) $(2a^2b^{-1}) \cdot (-3a^{-1}b^2) = -6ab$

g) $(-2a^2b)^{-1} \times (3a^{-2}b)^2 = -\frac{9b}{2a^6}$ h) $\left(\frac{3a^{-2}b}{4ab^{-2}}\right)^{-2} = \frac{16a^6}{9b^6}$ i) $\left(\frac{-2a^2b^{-1}}{3a}\right)^{-1} \cdot \left(\frac{3a^{-2}b}{2a}\right)^{-2} = \frac{-2a^5}{3b}$

7. Write the following expressions as a power of 2.

a) $(2^4)^5 \times 16^3$ 2^{32} b) $16^2 \times 8^4 \times 4^0$ 2^{20}

c) $\frac{4^4 \times 2^2 \times 16}{2^5 \times 8^2}$ 2^3 d) $\frac{(2^3)^2 \times (4^2)^3}{(2^2)^3 \times 8^4}$ 2^0

8. Compare the numbers $A = (25)^4 \times 5^5$ and $B = (25^2 \times 125)^2 \times 5^0$ after expressing each of these numbers as a power of 5.

$A = 5^{13}$; $B = 5^{14}$; Since $14 > 13$ then $B > A$.

9. Compare the numbers $A = (2^{-4} \times 4^3)^2 \times \frac{1}{8}$ and $B = \frac{1}{32} \times (2^{-4} \times 8^2)^2$ after expressing each of these numbers as a power of 2.

$A = 2^1$; $B = 2^{-1}$; Since $1 > -1$ then $A > B$.

10. Simplify the following expressions using the laws of exponents. ($a \neq 0$)

a) $a^{2n} \times a^n$ ***a^{3n}*** **b)** $a^{2n-1} \times a^{3-n}$ ***a^{n+2}*** **c)** $a^{3n+1} \div a^{2n-1}$ ***a^{n+2}***

d) $(a^{2n+1})^2$ ***a^{4n+2}*** **e)** $\frac{a^{3n-2}}{a^{n+2}}$ ***a^{2n-4}*** **f)** $(a^{n+1} \times a^{n-1})^2$ ***a^{4n}***

11. Simplify the following expressions using the laws of exponents. ($a \neq 0$)

a) $(a^{n+1})^3 \times (a^{n-1})^2$ ***a^{5n+1}*** **b)** $\frac{(a^{2n+1})^3}{(a^{2n-1})^2}$ ***a^{2n+5}***

c) $\frac{(a^{n-1})^3 \times (a^{2n+1})^2}{a^{5n+4}}$ ***a^{2n-5}*** **d)** $\frac{(a^{n-1})^2 \times (a^{n+1})^3}{a^{3n+1} \times a^{2n-1}}$ ***a***

12. Express $(4^{2n+4} \times 2^{4n-5}) \div 16^{2n}$ as a power of 2.

2^3

13. If $x = a^3$ and $y = 3a^2$, write the following expressions in terms of a.

a) $3x^2y$ ***$9a^8$*** **b)** x^2y^3 ***$27a^{12}$***

c) $(2x^3y)^2$ ***$36a^{22}$*** **d)** $4xy^2 \div 2xy$ ***$6a^2$***

e) $3x^2y^4 \div (2xy)^2$ ***$\frac{27a^4}{4}$*** **f)** $\left(\frac{2xy^2}{3xy}\right)^2$ ***$4a^4$***

g) $x^{-2}y^2$ ***$9a^{-2}$*** **h)** $9x^2y^{-2}$ ***a^2***

14. The population of a city is given by $P = (1.02)^t$, where P is expressed in millions of people and t represents the number of years since 2005.

a) What will the population of this city be in 2010? ***Approximately 1 104 081 people.***

b) What was the population of this city in 2000? ***Approximately 905 731 people.***

15. In 2005, an employee of a company earned an annual salary of \$30 000. This company gives an annual 4 % raise to its employees. The equation $s = 30(1.04)^t$ expresses the employee's salary s (in thousands of dollars) as a function of the number of years t since 2005.

a) What was the employee's salary in

1. 2004? ***\$28 846*** 2. 2000? ***\$24 658***

b) What will the employee's salary be in

1. 2010? ***\$36 500*** 2. 2015? ***\$44 407***

1.2 Binary system

ACTIVITY 1 Decomposition of a number into a sum of powers

a) Raphael must empty 27 litres of water into a minimum number of containers.

1. If Raphael can only use 10 l or 1 l containers, how many containers of each kind will he use? ***Two 10 litre containers and seven 1 litre containers.***

2. If Raphael can only use 16 l, 8 l, 4 l, 2 l, or 1 l containers, how many containers of each kind will he use?

   One 16 litre container, one 8 litre container, one 2 litre container and one 1 litre container.

The numbers 16, 8, 4, 2 and 1 are powers of 2.

b) Any natural number can be decomposed into a **sum of powers** of any natural number greater than 1.

For example, the number 31 can be decomposed into a sum of powers of 2.

$31 = 16 + 15$ (The largest power of 2 going into 31 is 16 (2^4))
$= 16 + (8 + 7)$ (The largest power of 2 going into 15 is 8 (2^3))
$= 16 + 8 + (4 + 3)$ (The largest power of 2 going into 7 is 4 (2^2))
$= 16 + 8 + 4 + (2 + 1)$ (The largest power of 2 going into 3 is 2 (2^1))
$= 2^4 + 2^3 + 2^2 + 2^1 + 2^0$ Note that $1 = 2^0$.

Decompose the number 100 into

1. a sum of powers of 2.

   $100 = 64 + 36$
   $= 64 + (32 + 4)$
   $= 2^6 + 2^5 + 2^2$

2. a sum of powers of 3.

   $100 = 81 + 19$
   $= 81 + (9 + 10)$
   $= 81 + 9 + 9 + 1$
   $= 3^4 + 2 \times 3^2 + 3^0$

SYSTEM IN BASE 2

Given a number composed of digits, the value of a digit depends on its **position** in the number as well as the **numeration system**.

– The **decimal** system, or system in base 10, uses the 10 digits 0, 1, 2, ..., 9.
In the decimal system, the value of a digit a in position n is: $a \times 10^n$.
Ex.: Consider, in the decimal system, the number 7445.
The following table gives the position and the value of each digit.

Digit	7	4	4	5
Position	3	2	1	0
Value	7×10^3	4×10^2	4×10^1	5×10^0

Note that the positions are numbered from right to left: 0, 1, 2, ...

The decomposition of the number 7445 into a sum of powers of 10 is:

$7445 = 7 \times 10^3 + 4 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$.

– The binary system, or system in base 2, uses the digits 0 and 1.
In the binary system, the value of a digit a in position n is: $a \times 2^n$.
Ex.: Consider, in the binary system, the number 1101, written 1101_2.
The following table gives the position and the value of each digit.

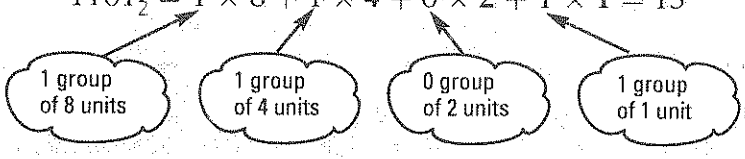
Digit	1	1	0	1
Position	3	2	1	0
Value	1×2^3	1×2^2	0×2^1	1×2^0

The decomposition of the number 1101_2 into a sum of powers of 2 is:

$$1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

By doing the sum of these powers of 2, we get the equivalent to the number 1101_2 in the decimal system.

$$1101_2 = 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 = 13$$



1. **a)** Expand the following numbers into a sum of powers of 2.

1. 27 = $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^1 + 1 \times 2^0$
2. 22 = $1 \times 2^4 + 1 \times 2^2 + 1 \times 2^1$
3. 49 = $1 \times 2^5 + 1 \times 2^4 + 1 \times 2^0$
4. 38 = $1 \times 2^5 + 1 \times 2^2 + 1 \times 2^1$

b) Write each of the preceding numbers in the binary system.

1. 27 = $11\ 011_2$
2. 22 = $10\ 110_2$
3. 49 = $110\ 001_2$
4. 38 = $100\ 110_2$

2. The following numbers are in the binary system. Write each number as a sum of powers of 2 and find the equivalent of each number in the decimal system.

a) 100_2: $1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4$

b) 101_2: $1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$

c) 1101_2: $1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13$

d) 1010_2: $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 10$

e) $11\ 101_2$: $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 29$

3. We must arrange 14 chocolate bars into a minimum of packages. Each package contains 1, 2, 4 or 8 bars. How many packages of each kind must we make?

1 package of 8 bars, 1 package of 4 bars, 1 package of 2 bars.

4. Find the minimum of quarters, nickels and pennies that are required to get an amount of

a) 118 ¢. *4 quarters, 3 nickels and 3 pennies*

b) 122 ¢. *4 quarters, 4 nickels and 2 pennies*

c) 134 ¢. *5 quarters, 1 nickel and 4 pennies*

d) 47 ¢. *1 quarter, 4 nickels and 2 pennies*

5. Write each of the preceding amounts in the form of a sum of powers of 5.

a) 118 $4 \times 5^2 + 3 \times 5^1 + 3 \times 5^0$ b) 122 $4 \times 5^2 + 4 \times 5^1 + 2 \times 5^0$

c) 134 $5 \times 5^2 + 1 \times 5^1 + 4 \times 5^0$ d) 47 $1 \times 5^2 + 4 \times 5^1 + 2 \times 5^0$

6. Write each of the preceding numbers in a system in base 5.

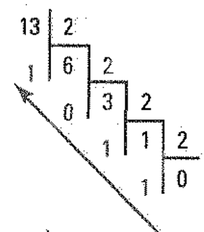
a) 118 433_5 b) 122 442_5 c) 134 514_5 d) 47 142_5

7. Write the first 10 natural numbers 0, 1, …, 9 in the binary system.

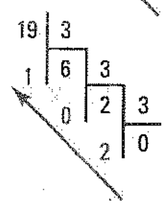
$0 = 0_2$; $1 = 1_2$; $2 = 10_2$; $3 = 11_2$; $4 = 100_2$; $5 = 101_2$; $6 = 110_2$; $7 = 111_2$; $8 = 1000_2$; $9 = 1001_2$

CONVERSION TO THE BINARY SYSTEM: DIVISION TECHNIQUE

- To convert a number from the decimal system into base 2, we perform many successive divisions by 2 until we get a zero quotient.
  Use the remainders, in the order indicated by the arrow, to write the number.
  Ex.: Converting 13 from the decimal system into base 2.
  We get: $13_{10} = 1101_2$.
  Verification: $13_{10} = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$



- To convert a number from the decimal system into base 3, we proceed in the same way by performing many successive divisions by 3.
  Ex.: Converting 19 from the decimal system into base 3.
  We get: $19_{10} = 201_3$.
  Verification: $19_{10} = 2 \times 3^2 + 0 \times 3^1 + 1 \times 3^0$



8. Convert the following numbers

1. into base 2. 2. into base 3. 3. into base 5.

a) 17 1. $10\,001_2$ 2. 122_3 3. 32_5

b) 35 1. $100\,011_2$ 2. 1022_3 3. 120_5

c) 72 1. $1\,001\,000_2$ 2. 2200_3 3. 242_5

9. Perform the following operations and express each answer in base 2.

a) $101_2 + 111_2$ 1100_2 b) $101_2 - 11_2 =$ 10_2

c) $110_2 \times 10_2$ 1100_2 d) $1010_2 \div 101_2 =$ 10_2

e) $100_2 \times 101_2$ 1001_2 f) $1001_2 - 100_2 =$ 101_2

1.3 Powers with rational or irrational exponent

Activity 1 n^{th} root of a real number

a) Complete.

1. $\sqrt[2]{64}$ = ***8*** because ***$8^2 = 64$***
2. $\sqrt[2]{-64}$ does not exist in ℝ because ***$-64 < 0$***
3. $\sqrt[3]{64}$ = ***4*** because ***$4^3 = 64$***
4. $\sqrt[3]{-64}$ = ***−4*** because ***$(-4)^3 = -64$***

b) Complete. $\sqrt[n]{a} = b \Leftrightarrow$ ***$b^n = a$***

c) Under what conditions does $\sqrt[n]{a}$ not exist in ℝ?

When n is even and a is negative.

---

n^{th} ROOT OF A REAL NUMBER

Given a natural number n and a real number a, the n^{th} root of the real number a, written $\sqrt[n]{a}$, is the unique real number b for which $b^n = a$.

$$\sqrt[n]{a} = b \Leftrightarrow b^n = a$$

Note that $\sqrt[n]{a}$ does not exist in ℝ when **n** is even and ***a*** is negative.

> n is called the index, a is the radicand, and $\sqrt{}$ is the radical.

By convention, we do not write the index 2. Thus, $\sqrt[2]{a} = \sqrt{a}$.

Ex.: $\sqrt[4]{16} = 2$ because $2^4 = 16$; $\sqrt[3]{8} = 2$ because $2^3 = 8$; $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$
$\sqrt[4]{-16} \notin$ ℝ because the index 4 is even and the radicand −16 is negative.

---

1. Determine, if possible.

a) $\sqrt{25}$ ***5*** **b)** $\sqrt{-25}$ $\notin$ ℝ **c)** $\sqrt[3]{27}$ ***3*** **d)** $\sqrt[3]{-27}$ ***−3***

e) $\sqrt[4]{81}$ ***3*** **f)** $\sqrt[4]{-81}$ $\notin$ ℝ **g)** $\sqrt{(-7)^2}$ ***7*** **h)** $\sqrt[3]{(-2)^3}$ ***−2***

2. Using a calculator, find the value of the following expressions (round your answer to the nearest thousandth).

a) $\sqrt{2}$ ***1.414*** **b)** $\sqrt{3}$ ***1.732*** **c)** $\sqrt[3]{2}$ ***1.26*** **d)** $\sqrt[3]{3}$ ***1.442***

3. True or false?

a) The equation $x^2 = 25$ has 2 real solutions 5 and −5. ***True***

b) The equation $x^2 = -25$ has no real solutions. ***True***

c) The equation $x^3 = 8$ has the number 2 as its only solution. ***True***

d) The equation $x^3 = -8$ has the number −2 as its only solution. ***True***

4. Simplify $\sqrt{x^2}$ if

a) x is positive. ***$\sqrt{x^2} = x$*** **b)** x is negative. ***$\sqrt{x^2} = -x$***

ACTIVITY 2 Powers with rational exponent

a) For any real number a, and any non-zero natural number n, we have: $a^{\frac{1}{n}} = \sqrt[n]{a}$.
Evaluate, if possible.

1. $16^{\frac{1}{4}}$ **2** 2. $(-16)^{\frac{1}{4}}$ $\notin \mathbb{R}$ 3. $8^{\frac{1}{3}}$ **2** 4. $(-8)^{\frac{1}{3}}$ **−2**

b) If $a^{\frac{1}{n}}$ exists in $\mathbb{R}$, then for any natural number m, we have: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$.
Evaluate, if possible.

1. $16^{\frac{3}{2}}$ **64** 2. $(-8)^{\frac{2}{3}}$ **4** 3. $(-16)^{\frac{3}{4}}$ $\notin \mathbb{R}$ 4. $64^{\frac{2}{3}}$ **16**

POWERS WITH RATIONAL EXPONENT

- **Definitions**
  - For any real number a, and any non-zero natural number n, we have:

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

Ex.: $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$; $(-64)^{\frac{1}{3}} = \sqrt[3]{-64} = -4$; $(-81)^{\frac{1}{4}} = \sqrt[4]{-81} \notin \mathbb{R}$

The power $a^{\frac{1}{n}}$ does not exist in $\mathbb{R}$ when n is even and a is negative.

  - If $a^{\frac{1}{n}}$ exists in $\mathbb{R}$, then for any natural number m, we have:

$$\boxed{a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m}$$

Ex.: $16^{\frac{3}{2}} = \left(\sqrt[2]{16}\right)^3 = 4^3 = 64$; $(-8)^{\frac{2}{3}} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$

- **Laws of exponents**

The five laws of exponents also apply when the exponents m and n are rational numbers.

Ex.: 1. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$ 2. $a^{\frac{3}{2}} \div a^{\frac{2}{3}} = a^{\frac{3}{2}-\frac{2}{3}} = a^{\frac{5}{6}}$

3. $(a \cdot b)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$ 4. $\left(\frac{a}{b}\right)^{\frac{3}{2}} = \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$ 5. $\left(a^{\frac{2}{3}}\right)^{\frac{3}{4}} = a^{\frac{1}{2}}$

5. Calculate the following real numbers.

a) $25^{\frac{3}{2}}$ **125** **b)** $(-27)^{\frac{2}{3}}$ **9** **c)** $16^{1.5}$ **64**

d) $9^{-\frac{3}{2}}$ $\frac{1}{27}$ **e)** $8^{-\frac{2}{3}}$ $\frac{1}{4}$ **f)** $4^{-2.5}$ $\frac{1}{32}$

6. **a)** If $a^{\frac{1}{n}}$ exists, show that $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$. $\left(\sqrt[n]{a}\right)^m = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

b) Calculate $4^{\frac{3}{2}}$ in two different ways. 1. $\left(\sqrt{4}\right)^3 = 2^3 = 8$ 2. $\sqrt{4^3} = \sqrt{64} = 8$

7. Apply the laws of exponents to simplify the following expressions.

a) $x^{\frac{2}{3}} \cdot x^{-\frac{1}{2}} =$ $x^{\frac{1}{6}}$ b) $(x^{-2})^{-\frac{3}{2}} =$ x^3 c) $\dfrac{x^{-\frac{2}{3}}}{x^{-\frac{3}{2}}} =$ $x^{\frac{5}{6}}$

d) $(x^{-2}y^3)^{\frac{1}{6}} =$ $\frac{y^{\frac{1}{2}}}{x^{\frac{1}{3}}}$ e) $\left(\dfrac{x^{-2}}{y^2}\right)^{-\frac{3}{2}} =$ x^3y^3 f) $(x^3y^{-1})^{\frac{2}{3}} \cdot (xy^2)^{\frac{1}{2}} =$ $x^{\frac{5}{2}}y^{\frac{1}{3}}$

g) $\left(\dfrac{x^2}{y^{-1}}\right)^{\frac{1}{2}} \cdot \left(\dfrac{x^3}{y}\right)^{\frac{1}{3}} =$ $x^2y^{\frac{1}{6}}$ h) $(4x^2y^4)^{\frac{1}{2}} =$ $2xy^2$ i) $\left(\dfrac{8x^3y^6}{27}\right)^{\frac{2}{3}} =$ $\frac{4x^2y^4}{9}$

8. Write the expression $\dfrac{4^3 \times \sqrt{8}}{8^2 \times \sqrt[3]{16}}$ as a power of 2. $2^{\frac{1}{6}}$

9. Write the expression $\dfrac{\sqrt{5} \times \sqrt[3]{25}}{\sqrt[6]{625}}$ as a power of 5. $5^{\frac{1}{2}}$

10. Given that $a > 0$, simplify the expression $\dfrac{a^2 \cdot \sqrt{a^3}}{\sqrt[3]{a^2} \cdot \sqrt{a^4}}$. $a^{\frac{5}{6}}$

ACTIVITY 3 Powers with irrational exponent

a) Calculate the following powers with rational exponents.

1. 2^3 8 2. 2^{-3} $\frac{1}{8}$ 3. $4^{\frac{1}{2}}$ 2 4. $27^{\frac{2}{3}}$ 9 5. 5^0 1

b) Use a calculator to calculate the following powers with irrational exponent (round to the nearest hundredth).

1. $2^{\sqrt{2}}$ 2.67 2. 2^{π} 8.82 3. $2^{-\pi}$ 0.11

POWERS OF A POSITIVE REAL NUMBER

- Given a positive real number a and a rational or irrational number x, the x^{th} power of the number a, written a^x, exists for any x.
  Ex.: Using a calculator, we get (rounded to the thousandth)
  $(\sqrt{2})^{\pi} = 2.971$; $\pi^{\sqrt{2}} = 5.047$
- The five laws of exponents also apply when the exponents are real numbers.
  Ex.: 1. $a^x \cdot a^y = a^{x+y}$ $(a \geq 0)$ 2. $\dfrac{a^x}{a^y} = a^{x-y}$ $(a > 0)$
  3. $(a \cdot b)^x = a^x \cdot b^x$ $(a \geq 0, b \geq 0)$ 4. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $(a \geq 0, b > 0)$
  5. $(a^x)^y = a^{xy}$ $(a \geq 0)$

11. Given $f(x) = 3 \times 4^x$. Calculate.

a) $f(0)$ 3 b) $f(2)$ 48 c) $f(-2)$ $\frac{3}{16}$ d) $f\left(\frac{1}{2}\right)$ 6 e) $f\left(-\frac{3}{2}\right)$ 0.375

12. Given $f(x) = -2 \times 9^x$. Calculate.

a) $f(0)$ -2 b) $f(1)$ -18 c) $f(-1)$ $-\frac{2}{9}$ d) $f\left(\frac{1}{2}\right)$ -6 e) $f\left(-\frac{1}{2}\right)$ $-\frac{2}{3}$

13. Given $f(x) = 3^x$. Use a calculator to determine (round to the nearest hundredth)

a) $f\left(\frac{1}{2}\right)$ 1.73 b) $f\left(-\frac{1}{2}\right)$ 0.58 c) $f(\sqrt{2})$ 4.73 d) $f(\pi)$ 31.54

1.4 Square root of a real number

Activity 1 Properties of radicals

a) Consider two real numbers a and b and the equality $\sqrt{a} = b$.

1. What conditions must we put on the real numbers a and b? ***$a \geqslant 0$ and $b \geqslant 0$***
2. Complete: $\sqrt{a} = b \Leftrightarrow$ ***$b^2 = a$***

b) True or false?

1. $\sqrt{4 \times 25} = \sqrt{4} \times \sqrt{25}$ ***True***
2. $\sqrt{\frac{100}{25}} = \frac{\sqrt{100}}{\sqrt{25}}$ ***True***
3. $\sqrt{16+9} = \sqrt{16} + \sqrt{9}$ ***False***
4. $\sqrt{25-9} = \sqrt{25} - \sqrt{9}$ ***False***

c) Justify the steps showing that $\sqrt{ab} = \sqrt{a} \times \sqrt{b} \, (a \in \mathbb{R}_+, b \in \mathbb{R}_+)$.

$\sqrt{ab} = (ab)^{\frac{1}{2}}$ ***Definition: $\sqrt[n]{x} = x^{\frac{1}{n}}$***

$= a^{\frac{1}{2}} b^{\frac{1}{2}}$ ***Laws of exponents: Power of a product***

$= \sqrt{a}\sqrt{b}$ ***Definition: $\sqrt[n]{x} = x^{\frac{1}{n}}$***

d) Justify the steps showing that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \, (a \in \mathbb{R}_+ \text{ and } b \in \mathbb{R}^*_+)$.

$\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$ ***Definition***

$= \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ ***Laws of exponents: Power of a quotient***

$= \frac{\sqrt{a}}{\sqrt{b}}$ ***Definition***

PROPERTIES OF RADICALS

- The square root of a zero or positive real number a, written $\sqrt{a}$, is the zero or positive real number b such that $b^2 = a$.

  If $a \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$, $\boxed{\sqrt{a} = b \Leftrightarrow b^2 = a}$ Ex.: $\sqrt{25} = 5$ because $5^2 = 25$

- If $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$, we have:

$\boxed{\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}}$		Ex.: $\sqrt{16 \times 9} = \sqrt{16} \times \sqrt{9}$
$\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$	$(b \neq 0)$	Ex.: $\sqrt{\frac{16}{100}} = \frac{\sqrt{16}}{\sqrt{100}}$
$\boxed{\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}}$	$(ab \neq 0)$	Ex.: $\sqrt{64+36} \neq \sqrt{64} + \sqrt{36}$
$\boxed{\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}}$	$(ab \neq 0)$	Ex.: $\sqrt{100-64} \neq \sqrt{100} - \sqrt{64}$

1. a) Is the square root of a positive number unique? *Yes*

b) Does the square root of a negative number exist in the set of real numbers? *No*

c) Can the square root of a number be negative? *No*

2. Calculate.

a) $\sqrt{25} \times \sqrt{64}$ *40* b) $\sqrt{8} \times \sqrt{2}$ *4* c) $(\sqrt{5})^2$ *5*

d) $\sqrt{\frac{100}{64}}$ $\frac{5}{4}$ e) $\frac{\sqrt{75}}{\sqrt{3}}$ *5* f) $\sqrt{72} \div \sqrt{2}$ *6*

g) $\sqrt{81+144}$ *15* h) $\sqrt{225-81}$ *12* i) $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{6}}$ *4*

3. The associative property of multiplication in the set ℝ of real numbers enables you to establish that:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd} \quad (b \geqslant 0, d \geqslant 0)$$

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}} \quad (b \geqslant 0, c > 0, d > 0)$$

Perform the following operations using this property.

a) $3\sqrt{2} \times \sqrt{5}$ $3\sqrt{10}$ b) $2 \times 5\sqrt{3}$ $10\sqrt{3}$ c) $-3\sqrt{2} \times 5\sqrt{3}$ $-15\sqrt{6}$

d) $12\sqrt{6} \div 4\sqrt{3}$ $3\sqrt{2}$ e) $12\sqrt{2} \div 2$ $6\sqrt{2}$ f) $12\sqrt{20} \div 4\sqrt{5}$ *6*

ACTIVITY 2 Reducing the radicand

Justify the steps which enable you to reduce the radicand.

$\sqrt{75} = \sqrt{25 \times 3}$ $75 = 25 \times 3$

$= \sqrt{25} \times \sqrt{3}$ $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

$= 5\sqrt{3}$ *Evaluation of* $\sqrt{a}$

REDUCING THE RADICAND

- Let us illustrate the procedure in the following example.

  Ex.: $\sqrt{80} = \sqrt{16 \times 5}$ The radicand is written as the product of 2 factors with one factor being a perfect square number.

  $= \sqrt{16} \times \sqrt{5}$ Apply the property $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.

  $= 4 \times \sqrt{5}$ Evaluate the square root of the square number factor.

  $= 4\sqrt{5}$
- The list of perfect square numbers less than 200 are:
  0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.

4. Write each of the following numbers in the form $a\sqrt{b}$, where b is the smallest possible integer.

a) $\sqrt{32}$ $4\sqrt{2}$ b) $\sqrt{98}$ $7\sqrt{2}$ c) $\sqrt{180}$ $6\sqrt{5}$

d) $\sqrt{720}$ $12\sqrt{5}$ e) $\sqrt{500}$ $10\sqrt{5}$ f) $2\sqrt{40}$ $4\sqrt{10}$

g) $\sqrt{567}$ $9\sqrt{7}$ h) $\sqrt{1944}$ $18\sqrt{6}$ i) $\frac{3}{5}\sqrt{250}$ $3\sqrt{10}$

5. The distributive property of multiplication over addition in the set $\mathbb{R}$ of real numbers enables you to establish that: $a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}$ $(b \geqslant 0)$. Perform the following operations using this property.

a) $2\sqrt{5} + 4\sqrt{5}$ ***$6\sqrt{5}$***

b) $7\sqrt{2} - 2\sqrt{2}$ ***$5\sqrt{2}$***

c) $5\sqrt{18} - 6\sqrt{2}$ ***$9\sqrt{2}$***

d) $-3\sqrt{18} + 4\sqrt{8}$ ***$-\sqrt{2}$***

e) $2\sqrt{45} - 2\sqrt{28} + 3\sqrt{20} + 3\sqrt{63}$ ***$12\sqrt{5} + 5\sqrt{7}$***

f) $2\sqrt{75} - 2\sqrt{108} + 5\sqrt{48} - \sqrt{27} + 3\sqrt{12}$ ***$21\sqrt{3}$***

g) $\sqrt{3} - \frac{\sqrt{3}}{3}$ ***$\frac{2\sqrt{3}}{3}$***

h) $3\sqrt{20} - 2\sqrt{12} + \sqrt{45} + 4\sqrt{27}$ ***$8\sqrt{3} + 9\sqrt{5}$***

6. Perform the following operations.

a) $(-2\sqrt{5})^2$ ***20***

b) $3\sqrt{5} \times -2\sqrt{3}$ ***$-6\sqrt{15}$***

c) $4\sqrt{3} \times 2\sqrt{15}$ ***$24\sqrt{5}$***

d) $5\sqrt{18} - 6\sqrt{2}$ ***$9\sqrt{2}$***

e) $2\sqrt{3}(5\sqrt{3} + \sqrt{5})$ ***$30 + 2\sqrt{15}$***

f) $(4\sqrt{5} + 3)(4\sqrt{5} - 3)$ ***71***

g) $3\sqrt{2}(\sqrt{2} + 1)$ ***$6 + 3\sqrt{2}$***

h) $2\sqrt{3}(2\sqrt{3} - \sqrt{2})$ ***$12 - 2\sqrt{6}$***

i) $5\sqrt{18} - 4\sqrt{50}$ ***$-5\sqrt{2}$***

j) $5\sqrt{8} - 2\sqrt{27} + 2\sqrt{75} - 3\sqrt{18}$ ***$\sqrt{2} + 4\sqrt{3}$***

ACTIVITY 3 Rationalizing the denominator

a) 1. Given a rational number x. Show that $(\sqrt{x})^2$ is rational.

$(\sqrt{x})^2 = (x^{\frac{1}{2}})^2 = x$ which is rational.

2. Consider the expression $\frac{a}{\sqrt{b}}$ where $b \in \mathbb{Q}^*$. Justify the steps which enable you to make the denominator a rational number.

$\frac{a}{\sqrt{b}} = \frac{a \cdot \sqrt{b}}{\sqrt{b} \cdot \sqrt{b}}$ ***We obtain an equivalent fraction by multiplying the numerator and denominator by the same number $\sqrt{b}$.***

$= \frac{a\sqrt{b}}{(\sqrt{b})^2}$ ***Multiply in the numerator and denominator.***

$= \frac{a\sqrt{b}}{b}$ ***$(\sqrt{b})^2 = b$. See 1.***

b) Use the identity $(a + b)(a - b) = a^2 - b^2$ to expand the following products.

1. $(\sqrt{5} + 2)(\sqrt{5} - 2)$ ***1***

2. $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$ ***4***

3. $(2 - \sqrt{3})(2 + \sqrt{3})$ ***1***

4. $(2\sqrt{3} + 1)(2\sqrt{3} - 1)$ ***11***

c) The expression $(a - b)$ is called the **conjugate** of the expression $(a + b)$.

1. What is the conjugate of $(3 + \sqrt{2})$? ***$3 - \sqrt{2}$***

2. Verify that the product of $(3 + \sqrt{2})$ by its conjugate is a rational number.

$(3 + \sqrt{2})(3 - \sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7$

d) Consider the expression $\frac{7}{\sqrt{5} + \sqrt{2}}$. Justify the steps which enable you to make the denominator a rational number.

$\frac{7}{\sqrt{5} + \sqrt{2}} = \frac{7(\sqrt{5} - \sqrt{2})}{(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})}$ ***Multiply the numerator and denominator by the denominator's conjugate. We get an equivalent fraction.***

$= \frac{7(\sqrt{5} - \sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2}$ ***Apply the identity $(a + b)(a - b) = a^2 - b^2$.***

$= \frac{7(\sqrt{5} - \sqrt{2})}{3}$ ***Reduce the denominator.***

RATIONALIZING THE DENOMINATOR

Rationalizing the denominator of an irrational expression consists of determining an equivalent expression with a rational denominator.

There are 2 cases:

1. To rationalize the expression $\frac{a}{\sqrt{b}}$, we multiply the numerator and denominator by the denominator.

$$\frac{a}{\sqrt{b}} = \frac{a\cdot\sqrt{b}}{\sqrt{b}\cdot\sqrt{b}} = \frac{a\sqrt{b}}{\left(\sqrt{b}\right)^2} = \frac{a\sqrt{b}}{b}$$

Ex.: $\frac{2}{\sqrt{3}} = \frac{2\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2} = \frac{2\sqrt{3}}{3}$

2. To rationalize the expression $\frac{a}{\sqrt{b}+\sqrt{c}}$, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a\left(\sqrt{b}-\sqrt{c}\right)}{\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt{b}-\sqrt{c}\right)} = \frac{a\left(\sqrt{b}-\sqrt{c}\right)}{\left(\sqrt{b}\right)^2-\left(\sqrt{c}\right)^2} = \frac{a\left(\sqrt{b}-\sqrt{c}\right)}{b-c}$$

Ex.: $\frac{1}{\sqrt{5}+\sqrt{3}} = \frac{\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)} = \frac{\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2} = \frac{\sqrt{5}-\sqrt{3}}{5-3} = \frac{\sqrt{5}-\sqrt{3}}{2}$

Note that the conjugate of $(a+b)$ is $(a-b)$ and that $(a+b)(a-b) = a^2 - b^2$.

7. Rationalize the denominator of the following expressions.

a) $\frac{3}{\sqrt{5}}$ $\frac{3\sqrt{5}}{5}$ b) $\frac{5}{\sqrt{10}}$ $\frac{\sqrt{10}}{2}$ c) $\frac{1}{\sqrt{2}}$ $\frac{\sqrt{2}}{2}$ d) $\frac{2}{3\sqrt{5}}$ $\frac{2\sqrt{5}}{15}$

e) $\frac{\sqrt{2}}{\sqrt{3}}$ $\frac{\sqrt{6}}{3}$ f) $\sqrt{\frac{2}{5}}$ $\frac{\sqrt{10}}{5}$ g) $\frac{1+\sqrt{2}}{\sqrt{5}}$ $\frac{\sqrt{5}+\sqrt{10}}{5}$ h) $\frac{1-\sqrt{3}}{\sqrt{3}}$ $\frac{\sqrt{3}-3}{3}$

8. For each of the following expressions,

1. find the conjugate expression. 2. multiply the expression by its conjugate.

a) $\sqrt{3}+1$ 1. $\sqrt{3}-1$ 2. $\left(\sqrt{3}\right)^2 - 1^2 = 2$

b) $\sqrt{5}-2$ 1. $\sqrt{5}+2$ 2. $\left(\sqrt{5}\right)^2 - 2^2 = 1$

c) $\sqrt{3}+\sqrt{2}$ 1. $\sqrt{3}-\sqrt{2}$ 2. $\left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2 = 1$

d) $2\sqrt{3}-3$ 1. $2\sqrt{3}+3$ 2. $\left(2\sqrt{3}\right)^2 - 3^2 = 3$

9. Rationalize the denominator of the following expressions.

a) $\frac{1}{\sqrt{6}+2}$ $\frac{\sqrt{6}-2}{2}$ b) $\frac{\sqrt{3}}{\sqrt{5}-1}$ $\frac{\sqrt{15}+\sqrt{3}}{4}$ c) $\frac{3}{\sqrt{5}-\sqrt{3}}$ $\frac{3\sqrt{5}+3\sqrt{3}}{2}$

d) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ $5+2\sqrt{6}$ e) $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$ $\frac{30-12\sqrt{6}}{6}$ f) $\frac{\sqrt{2}}{5\sqrt{2}-2\sqrt{5}}$ $\frac{5+\sqrt{10}}{15}$

10. Rationalize the numerator of the following expressions.

a) $\frac{\sqrt{3}+1}{2}$ $\frac{1}{\sqrt{3}-1}$ b) $\frac{\sqrt{5}+\sqrt{2}}{6}$ $\frac{1}{2\left(\sqrt{5}-\sqrt{2}\right)}$ c) $\frac{3\sqrt{2}-\sqrt{3}}{3\sqrt{2}+\sqrt{3}}$ $\frac{5}{7+2\sqrt{3}}$

11. Rationalize the numerator of the expression $\frac{\sqrt{x+h}-\sqrt{x}}{h}$. $(h \neq 0)$

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

12. Rationalize the denominator of the following expressions.

a) $\frac{1}{\sqrt{x+1}+\sqrt{x}}$ $\quad \sqrt{x+1}-\sqrt{x}$

b) $\frac{x-4}{(x-1)-\sqrt{2x+1}}$ $(x \neq 4)$ $\quad \frac{x-1+\sqrt{2x+1}}{x}$

c) $\frac{x^2-3x}{x-1-\sqrt{x+1}}$ $(x \neq 0$ and $x \neq 3)$ $\quad x-1+\sqrt{x+1}$

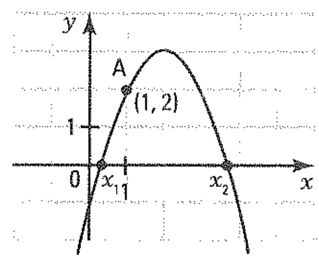
13. Consider the second degree equation $x^2 - 2x - 1 = 0$ in the form $ax^2 + bx + c = 0$.

a) Determine the solutions x_1 and x_2 of this equation. $\quad x_1 = 1-\sqrt{2};\ x_2 = 1+\sqrt{2}$

b) Verify the following properties.

1. $x_1 + x_2 = \frac{-b}{a}$. $\quad x_1 + x_2 = 2$ *and* $-\frac{b}{a} = 2$
2. $x_1 \cdot x_2 = \frac{c}{a}$. $\quad x_1 \cdot x_2 = (1+\sqrt{2})(1-\sqrt{2}) = -1$ *and* $\frac{c}{a} = -1$

14. The quadratic function f on the right has two zeros x_1 and x_2. If $x_1 = 2-\sqrt{3}$, $x_2 = 2+\sqrt{3}$ and $f(1) = 2$, determine



a) the rule of function f.

$f(x) = a(x - x_1)(x - x_2)$

$f(x) = a(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$

$f(x) = a(x^2 - 4x + 1)$

$f(1) = 2 \Rightarrow a = -1$. *Thus* $f(x) = -x^2 + 4x - 1$

b) max f. $\quad 3$

15. For what values of x are the following expressions defined?

a) $\sqrt{x-2}$ $\quad x \in [2, +\infty[$

b) $\sqrt[3]{x-2}$ $\quad x \in \mathbb{R}$

c) $\sqrt{2x-6}$ $\quad x \in [3, +\infty[$

d) $\sqrt{-3x+6}$ $\quad x \in]-\infty, 2]$

e) $\frac{1}{\sqrt{x-2}}$ $\quad x \in]2, +\infty[$

f) $\frac{x-1}{\sqrt{x-1}}$ $\quad x \in]1, +\infty]$

16. For what values of x are the following expressions defined?

a) $\sqrt{x^2-9}$ $\quad x \in]-\infty, -3] \cup [3, +\infty[$

b) $\sqrt{x^2+1}$ $\quad x \in \mathbb{R}$

c) $\sqrt{1-x^2}$ $\quad x \in [-1, 1]$

d) $\sqrt{x^2-5x+6}$ $\quad x \in]-\infty, 2] \cup [3, +\infty[$

17. For what values of x are the following expressions defined?

a) $\sqrt{\frac{x-1}{x+2}}$ $\quad x \in]-\infty, -2[\cup [1, +\infty[$

b) $\sqrt{\frac{x^2-1}{x^2-9}}$ $\quad x \in]-\infty, -3[\cup [-1, 1] \cup]3, +\infty[$

1.5 Factoring a polynomial

ACTIVITY 1 Factoring a polynomial

Factor the following polynomials.

a) $15x^4 + 20x^3 - 10x^2$ ___ $5x^2(3x^2 + 4x - 2)$ ___

b) $6x^2 - 2xy + 9x - 3y$ ___ $(2x + 3)(3x - y)$ ___

c) $16x^2 - 9$ ___ $(4x + 3)(4x - 3)$ ___

d) $9x^2 + 12x + 4$ ___ $(3x + 2)^2$ ___

e) $x^2 - 8x + 15$ ___ $(x - 3)(x - 5)$ ___

f) $2x^2 + 7x + 3$ ___ $(2x + 1)(x + 3)$ ___

FACTORING A POLYNOMIAL

- **Factoring** a polynomial means writing the polynomial as a product of factors.
- **Removing a common factor** is a method which can be used to factor a polynomial composed of monomials which all have a common factor. To factor, you need to apply the distributive property of multiplication over addition.

factor

$$ab + ac = a(b + c)$$

expand

Ex.: $P(x) = 6x^4 + 15x^3 - 18x^2$

$= 3x^2(2x^2 + 5x - 6)$

- **Factoring by grouping** is a method which enables you to factor polynomials by grouping the terms which contain a common factor. You then remove the common factor in each of the groupings.

Ex.: Factor the following expression using factoring by grouping.

$9x^2 - 12xy^2 + 6xy - 8y^3$ ← Group the terms containing a common factor.

$= 3x(3x - 4y^2) + 2y(3x - 4y^2)$ ← Remove the common factor in each grouping.

$= (3x - 4y^2)(3x + 2y)$ ← Remove the common factor a 2nd time.

- **A difference of two squares** is factorable.

$$a^2 - b^2 = (a + b)(a - b)$$

Ex.: $9x^2 - 4y^2 = (3x + 2y)(3x - 2y)$

- **Perfect square trinomials** are factorable.

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Ex.: $4x^2 + 12x + 9 = (2x + 3)^2$

$9x^2 - 30xy + 25y^2 = (3x - 5y)^2$

- The "**product and sum**" method enables you to factor a second degree trinomial. Let us illustrate this method by factoring $P(x) = 2x^2 + 7x + 6$.

1. **Identify** the coefficients a, b and c.	1. $a = 2$; $b = 7$; $c = 6$
2. **Find** two integers m and n such that $\begin{cases} m \cdot n = ac \leftarrow \text{product of the end coefficients} \\ m + n = b \leftarrow \text{middle coefficient.} \end{cases}$	2. $\begin{cases} mn = 12 \\ m + n = 7 \end{cases}$ $m = 4, n = 3$
3. Write: $ax^2 + bx + c = ax^2 + mx + nx + c$ and **factor** by grouping.	3. $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$ $= 2x(x + 2) + 3(x + 2)$ $= (x + 2)(2x + 3)$

1. Factor the following polynomials.

a) $6x^3y^3 - 9x^3y + 6x^2y^2$ *$3x^2y(2xy^2 - 3x + 2y)$*

b) $25x^2 - 9y^2$ *$(5x + 3y)(5x - 3y)$*

c) $2x^2 + 3xy - 10x - 15y$ *$(2x + 3y)(x - 5)$*

d) $x^3 + x^2 + x - 3$ *$(x^2 + 1)(x - 3)$*

e) $16x^2 - (x + 2)^2$ *$(5x + 2)(3x - 2)$*

f) $(x + 1)^2 - (2x + 3)^2$ *$(3x + 4)(-x - 2)$*

g) $4x^2 - 4x + 1$ *$(2x - 1)^2$*

h) $9x^2 + 12xy + 4y^2$ *$(3x + 2y)^2$*

i) $x^2 - 8x + 15$ *$(x - 3)(x - 5)$*

j) $2x^2 - 7x + 3$ *$(2x - 1)(x - 3)$*

k) $2x^2 - 9x + 4$ *$(2x - 1)(x - 4)$*

l) $3x^2 - 4x - 4$ *$(3x + 2)(x - 2)$*

2. Factor the following polynomials completely.

a) $2x^2 - 50x$ *$2x(x + 5)(x - 5)$*

b) $12x^3 - 12x^2 + 3x$ *$3x(2x - 1)^2$*

c) $4x^3 - 4x^2 - 8x$ *$4x(x + 1)(x - 2)$*

d) $x^4 - 81$ *$(x^2 + 9)(x + 3)(x - 3)$*

e) $(x^2 - 4) + (x - 2)^2$ *$2x(x - 2)$*

f) $x^4 - 8x^2 + 16$ *$(x + 2)^2(x - 2)^2$*

ACTIVITY 2 Factoring a second degree trinomial: Method of completing the square.

a) Fill in the missing term to obtain a perfect square trinomial and then factor.

1. $x^2 + 6x + \underline{9} = \underline{(x + 3)^2}$

2. $x^2 - 5x + \underline{\frac{25}{4}} = \underline{\left(x - \frac{5}{2}\right)^2}$

3. $x^2 + \frac{5}{2}x + \underline{\frac{25}{16}} = \underline{\left(x + \frac{5}{4}\right)^2}$

4. $x^2 - \frac{7}{3}x + \underline{\frac{49}{36}} = \underline{\left(x - \frac{7}{6}\right)^2}$

b) Justify the steps which enable you to factor a trinomial of the form $ax^2 + bx + c$.

Steps	Justifications
$P(x) = 2x^2 + 5x + 2$	$a = 2, b = 5, c = 2$
$= 2\left(x^2 + \frac{5}{2}x + 1\right)$	*Factor out the coefficient a = 2.*
$= 2\left(x^2 + \frac{5}{2}x + \ldots + 1 - \ldots\right)$	
$= 2\left[x^2 + \frac{5}{2}x + \frac{25}{16} + 1 - \frac{25}{16}\right]$	*Complete the perfect square trinomial.*
$= 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{9}{16}\right]$	*Factor the perfect square trinomial.*
$= 2\left(x + \frac{5}{4} + \frac{3}{4}\right)\left(x + \frac{5}{4} - \frac{3}{4}\right)$	*Factor the difference of two squares.*
$= 2(x + 2)\left(x + \frac{1}{2}\right)$	*Simplify the expression.*
$= (x + 2)(2x + 1)$	*Write as a product of two factors.*

FACTORING A SECOND DEGREE TRINOMIAL: METHOD OF COMPLETING THE SQUARE

The method of completing the square is illustrated below by factoring $P(x) = 2x^2 + 7x + 6$.

$P(x) = 2x^2 + 7x + 6$

$= 2\left(x^2 + \frac{7}{2}x + 3\right)$ ← Factor out the coefficient $a = 2$.

$= 2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) + 3 - \frac{49}{16}\right]$ ← Complete the perfect square trinomial.

$\frac{49}{16}$ is the square of half of the coefficient $\frac{7}{2}$ of the middle term.

$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{1}{16}\right]$ ← Factor the perfect square trinomial.

$= 2\left[\left(x + \frac{7}{4} + \frac{1}{4}\right)\left(x + \frac{7}{4} - \frac{1}{4}\right)\right]$ ← Factor the difference of two squares.

$= 2(x + 2)\left(x + \frac{3}{2}\right)$ ← Simplify.

$= (x + 2)(2x + 3)$ ← Write as a product of two binomials.

3. Factor the following second degree trinomials by completing the square.

a) $x^2 - 10x + 21$ ***(x - 7)(x - 3)***

b) $x^2 - 5x - 14$ ***(x - 7)(x + 2)***

c) $x^2 - 7x + 12$ ***(x - 3)(x - 4)***

d) $x^2 - 9x + 20$ ***(x - 5)(x - 4)***

e) $2x^2 + 7x + 3$ ***(2x + 1)(x + 3)***

f) $3x^2 + 5x - 2$ ***(3x - 1)(x + 2)***

g) $6x^2 + x - 2$ ***(3x + 2)(2x - 1)***

h) $10x^2 - 19x + 6$ ***(5x - 2)(2x - 3)***

ACTIVITY 3 Factoring a second degree trinomial: The roots method.

a) Justify the steps which enable you to write a second degree trinomial in standard form.

$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ ***Factor out a.***

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$ ***Complete the perfect square trinomial.***

$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$ ***Factor the perfect square trinomial.***

$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$ $\Delta = b^2 - 4ac$

b) From the standard form $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$,

1. show that when Δ is positive, the trinomial $ax^2 + bx + c$ is factorable into a product of two 1st degree factors, that is to say $ax^2 + bx + c = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the roots of the trinomial.

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] = a\left[\left(x + \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a}\right)\right]$$

$$= a\left(x - \frac{-b - \sqrt{\Delta}}{2a}\right)\left(x - \frac{-b + \sqrt{\Delta}}{2a}\right)$$

$$= a(x - x_1)(x - x_2) \text{ where } x_1 = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

2. Explain why the trinomial $ax^2 + bx + c$ is not factorable in $\mathbb{R}$ when Δ is negative.

$a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$ *cannot be factored since the expression* $\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}$ *is a sum of two squares which cannot be factored in* $\mathbb{R}$.

3. Factor the trinomial $ax^2 + bx + c$ when Δ is zero.

$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$.

FACTORING A SECOND DEGREE TRINOMIAL: THE ROOTS METHOD

The second degree trinomial $ax^2 + bx + c$ is factorable in $\mathbb{R}$ if and only if the discriminant Δ is positive or zero.

$$\boxed{\Delta \geqslant 0: ax^2 + bx + c = a(x - x_1)(x - x_2)}, \Delta = b^2 - 4ac$$

where $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$ are the roots of the trinomial.

$$\boxed{\Delta < 0: ax^2 + bx + c \text{ cannot be factored in } \mathbb{R}.}$$

Ex.: Factor $2x^2 + 7x + 6$.

$a = 2, b = 7, c = 6; \Delta = 1; x_1 = -2, x_2 = -\frac{3}{2}$.

$2x^2 + 7x + 6 = 2(x + 2)\left(x + \frac{3}{2}\right)$

$= (x + 2)(2x + 3)$

Ex.: Factor $4x^2 - 12x + 9$.

$a = 4, b = -12, c = 9; \Delta = 0; x_1 = x_2 = \frac{3}{2}$.

$4x^2 - 12x + 9 = 4\left(x - \frac{3}{2}\right)^2$

Ex.: $x^2 + x + 1$ is not factorable in $\mathbb{R}$ since $\Delta < 0$.
Indeed, $a = 1, b = 1, c = 1$ and $\Delta = -3$.

4. For each of the following trinomials, indicate if the trinomial is factorable.

a) $2x^2 + 3x + 1$ *Yes*

b) $2x^2 - x - 6$ *Yes*

c) $x^2 - x + 1$ *No*

d) $4x^2 - 28x + 49$ *Yes*

e) $x^2 + 2x - 1$ *Yes*

f) $4x^2 - 12x + 10$ *No*

5. Factor the following trinomials using the roots method.

a) $2x^2 + 13x + 15$ $(2x + 3)(x + 5)$

b) $4x^2 + 12x + 9$ $4\left(x + \frac{3}{2}\right)^2 = (2x + 3)^2$

c) $-4x^2 + 4x - 1$ $-4\left(x - \frac{1}{2}\right)^2 = -(2x - 1)^2$

d) $4x^2 + 4x - 3$ $4\left(x + \frac{3}{2}\right)\left(x - \frac{1}{2}\right) = (2x + 3)(2x - 1)$

e) $x^2 + 2x - 1$ $\left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right)$

f) $2x^2 + 4x - 4$ $2\left(x + 1 + \sqrt{3}\right)\left(x + 1 - \sqrt{3}\right)$

g) $6x^2 - 7x + 2$ $6\left(x - \frac{1}{2}\right)\left(x - \frac{2}{3}\right) = (2x - 1)(3x - 2)$

h) $8x^2 + 2x - 1$ $8\left(x - \frac{1}{4}\right)\left(x + \frac{1}{2}\right) = (4x - 1)(2x + 1)$

6. Simplify the following rational expressions after indicating the restrictions.

a) $\dfrac{x^2-3x+2}{(x-1)^2} = \dfrac{(x-1)(x-2)}{(x-1)^2}$

$= \dfrac{(x-2)}{(x-1)}\ (x \neq 1)$

b) $\dfrac{x^2-9}{x^2-8x+15} = \dfrac{(x+3)(x-3)}{(x-3)(x-5)}$

$= \dfrac{(x+3)}{(x-5)}\ (x \neq 3,\ x \neq 5)$

c) $\dfrac{2x^2-3x-2}{2x^2-5x-3} = \dfrac{(2x+1)(x-2)}{(2x+1)(x-3)}$

$= \dfrac{x-2}{x-3}\left(x \neq -\dfrac{1}{2} \text{ and } x \neq 3\right)$

d) $\dfrac{2x^2-12x^2+18x}{2x^3-18x} = \dfrac{2x(x-3)^2}{2x(x+3)(x-3)}$

$= \dfrac{x-3}{x+3}(x \neq 0,\ x \neq -3,\ x \neq 3)$

7. Solve the following equations.

a) $2x^2 - 9x - 5 = 0$

$S = \left\{-\dfrac{1}{2}, 5\right\}$

b) $9x^2 - 12x + 4 = 0$

$S = \left\{\dfrac{2}{3}\right\}$

c) $x^2 + 2x + 3 = 0$

$S = \varnothing$

d) $x^3 - x = 0$

$S = \{-1, 0, 1\}$

e) $x^2 + 4x + 1 = 0$

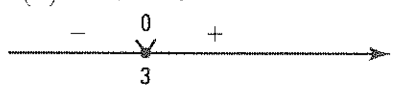
$S = \left\{-2-\sqrt{3}, -2+\sqrt{3}\right\}$

f) $-4x^2 + 4x - 1 = 0$

$S = \left\{\dfrac{1}{2}\right\}$

8. Determine the sign of the following polynomials.

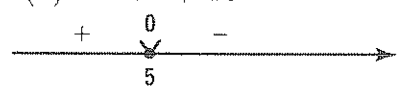
a) $P(x) = 2x - 6$



$P(x) \leq 0$ if $x \in]-\infty, 3]$

$P(x) \geq 0$ if $x \in [3, +\infty[$

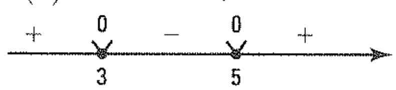
b) $P(x) = -2x^2 + 10$



$P(x) \geq 0$ if $x \in]-\infty, 5]$

$P(x) \leq 0$ if $x \in [5, +\infty[$

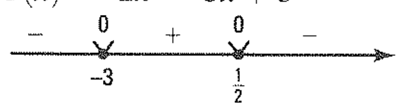
c) $P(x) = x^2 - 8x + 15$



$P(x) \leq 0$ if $x \in [3, 5]$

$P(x) \geq 0$ if $x \in]-\infty, 3] \cup [5, +\infty[$

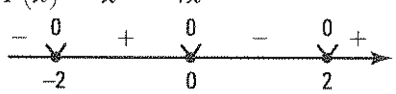
d) $P(x) = -2x^2 - 5x + 3$



$P(x) \leq 0$ if $x \in \left]-\infty, -3\right] \cup \left[\dfrac{1}{2}, +\infty\right[$

$P(x) \geq 0$ if $x \in \left[-3, \dfrac{1}{2}\right]$

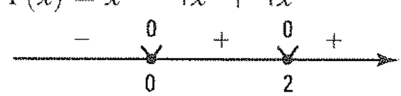
e) $P(x) = x^3 - 4x$



$P(x) \leq 0$ if $x \in]-\infty, -2] \cup [0, 2]$

$P(x) \geq 0$ if $x \in [-2, 0] \cup [2, +\infty[$

f) $P(x) = x^3 - 4x^2 + 4x$



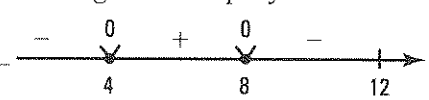
$P(x) \leq 0$ if $x \in]-\infty, 0]$

$P(x) \geq 0$ if $x \in [0, +\infty[$

9. The polynomial $R = -t^2 + 12t - 32$ gives the net revenue R, in thousands of dollars, of a bakery t months after opening ($0 \leq t \leq 12$). Determine and interpret the sign of this polynomial.

The bakery makes a profit if $t \in]4, 8[$.

The bakery has a deficit if $t \in [0, 4[\cup]8, 12]$.



Evaluation 1

1. Write, as a power of 2, the expression: $\dfrac{16^3 \times 2^{-4} \times \sqrt{24}}{\sqrt{6} \times 8^{\frac{3}{2}}}$. $2^{\frac{9}{2}}$

2. Simplify the following expressions.

a) $x^{-3} \cdot x^5$ x^2 b) $\left(x^{-\frac{2}{3}}\right)^{-6}$ x^4 c) $(x^2y^{-3})^2 \cdot (3x^2y)^{-1}$ $\frac{x^2}{3y^7}$

d) $\left(\dfrac{3x^{-2}}{2x^{-1}}\right)^2$ $\frac{9}{4x^2}$ e) $\sqrt{4x^8y^4}$ $2x^4y^2$ f) $\sqrt[3]{8x^6y^9}$ $2x^2y^3$

3. Reduce the radicand.

a) $\sqrt{72}$ $6\sqrt{2}$ b) $\sqrt{288}$ $12\sqrt{2}$

4. Perform the following operations.

a) $\sqrt{18} - \sqrt{12} - \sqrt{8} + 2\sqrt{3}$ $\sqrt{2}$ b) $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$ -6

5. Rationalize the denominator.

a) $\dfrac{2\sqrt{3}}{\sqrt{5}}$ $\frac{2\sqrt{15}}{5}$ b) $\dfrac{\sqrt{2}}{\sqrt{5} - \sqrt{3}}$ $\frac{\sqrt{10} + \sqrt{6}}{2}$ c) $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$ $\frac{7 - 2\sqrt{10}}{3}$

6. For what values of x are the following expressions defined?

a) $\sqrt{-2x + 6}$ $]-\infty, 3]$

b) $\sqrt{9 - x^2}$ $[-3, 3]$

c) $\sqrt{x^2 - 6x + 8}$ $]-\infty, 2] \cup [4, +\infty[$

d) $\sqrt{\dfrac{x^2 - 1}{16 - x^2}}$ $]-4, -1] \cup [1, 4[$

7. Write the number 28 of the decimal system

a) in base 2. $11\ 100_2$ b) in base 3. 1001_3 c) in base 5. 103_5

8. Convert the following numbers into the decimal system.

a) 1010_2 10 b) 211_3 22 c) 342_5 97

9. Factor the trinomial $2x^2 + 7x + 6$ by completing the square.

$$\begin{aligned}
2x^2 + 7x + 6 &= 2\left(x^2 + \frac{7}{2}x + \frac{49}{16} + 3 - \frac{49}{16}\right)\\
&= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{1}{16}\right]\\
&= 2\left[\left(x + \frac{7}{4} + \frac{1}{4}\right)\left(x + \frac{7}{4} - \frac{1}{4}\right)\right]\\
&= 2(x + 2)\left(x + \frac{3}{2}\right)\\
&= (x + 2)(2x + 3)
\end{aligned}$$

10. Factor the trinomial $x^2 + 2x - 1$.

$\Delta = 8;\ x_1 = -1 - \sqrt{2}\ ;\ x_2 = -1 + \sqrt{2}$

$x^2 + 2x - 1 = (x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

11. Factor the following polynomials completely.

a) $x^4 - 16$ $(x^2 + 4)(x + 2)(x - 2)$

b) $x^4 - 18x^2 + 81$ $(x + 3)^2(x - 3)^2$

c) $(x^2 + 1)^2 - 4x^2$ $(x + 1)^2(x - 1)^2$

d) $2x^5 - 4x^3 + 2x$ $2x(x + 1)^2(x - 1)^2$

12. Simplify the following rational expressions after indicating the restrictions.

a) $\dfrac{x^2 - 9}{x^2 + 8x + 15}$ $\dfrac{x - 3}{x + 5}$, $x \neq -3$ and $x \neq -5$

b) $\dfrac{2x^2 - 3x - 2}{x^2 - 7x + 10}$ $\dfrac{2x + 1}{x - 5}$, $x \neq 2$ and $x \neq 5$

13. Explain why the trinomial $x^2 + x + 1$ is not factorable in the set $\mathbb{R}$.

$\Delta = -3$ is negative.

14. For what values of m is the trinomial $mx^2 + (m + 1)x + m$ factorable in $\mathbb{R}$?

$\Delta = -3m^2 + 2m + 1;\ \Delta \geqslant 0 \Leftrightarrow m \in \dfrac{27x^3}{8y^3}$

15. For what values of m does the equation $mx^2 + mx + 1 = 0$ yield

a) two distinct solutions? $m \in]-\infty, 0[\cup]4, +\infty[$

b) only one solution? $m = 4$

c) no solution? $m \in [0, 4[$

16. The polynomial $R = 2t^2 - 18t + 36$ gives the net revenue R in thousands of dollars of a restaurant t months after the opening of a competing restaurant. $(0 \leqslant t \leqslant 12)$
Determine and interpret the sign of this polynomial.

The restaurant continues to make a profit for the first 3 months. The restaurant has a deficit over the next 6 months. The restaurant makes a profit again for the last 3 months.