Chapter 3

Real functions

CHALLENGE 3

EVALUATION 3

CHALLENGE 3

1. Determine the domain and range of the following functions.

a) $f(x) = -2|x - 3| + 1$
dom f = $\mathbb{R}$
ran f = $]-\infty, 1]$

b) $f(x) = -\sqrt{-x + 1} + 1$
dom f = $]-\infty, 1]$
ran f = $]-\infty, 1]$

c) $f(x) = \frac{2}{3(x - 1)} - 1$
dom f = $\mathbb{R} \setminus \{1\}$
ran f = $R \setminus \{-1\}$

2. Consider the functions $f(x) = 2x - 1$ and $g(x) = 3x^2 - 2x + 1$. Find the rule of

a) $g \circ f$
$g \circ f(x) = 3(2x - 1)^2 - 2(2x - 1) + 1$
$= 12x^2 - 16x + 6$

b) $f \circ g$
$f \circ g(x) = 2(3x^2 - 2x + 1) - 1$
$= 6x^2 - 4x + 1$

3. Determine the zeros of the following functions.

a) $f(x) = -2|x - 1| + 6$
−2 and 4

b) $f(x) = -2\sqrt{x - 3} + 6$
12

c) $f(x) = \frac{-3}{2(x + 1)} + 1$
$\frac{1}{2}$

4. What are the equations of the asymptotes of the function $f(x) = \frac{2}{5(x - 1)} - 4$?
The lines defined by the equations $x = 1$ and $y = -4$.

5. Study the sign of the following functions.

a) $f(x) = 4\left|-\frac{1}{2}(x - 1)\right| - 4$ $f(x) \leq 0$ if $x \in [-1, 3]$; $f(x) \geq 0$ if $x \in]-\infty, -1] \cup [3, +\infty[$

b) $f(x) = -2\sqrt{x + 3} + 4$ $f(x) \leq 0$ if $x \in [1, +\infty[$; $f(x) \geq 0$ if $x \in [-3, 1]$

c) $f(x) = \frac{4}{x - 3} + 2$ $f(x) \leq 0$ if $x \in [1, 3[$; $f(x) \geq 0$ if $x \in]-\infty, 1] \cup]3, +\infty[$

6. Describe the variation of the following functions.

a) $f(x) = -\frac{2}{3}|x - 2| + 4$ $f \nearrow$ over $]-\infty, 2]$; $f \searrow$ over $[2, +\infty[$

b) $f(x) = -\frac{1}{2}\sqrt{-2(x - 1)} + 1$ $f \nearrow$ over $]-\infty, 1]$

c) $f(x) = \frac{2}{x - 1} + 1$ $f \searrow$ over $\mathbb{R} \setminus \{1\}$

7. Find the rule of

a) an absolute value function whose graph has a vertex at V(−2, 6) and passes through the point A(1, −3). $y = -3|x + 2| + 6$

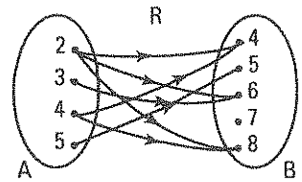
b) a rational function passing through the point A(3, 4) with asymptotes defined by the lines $x = 1$ and $y = 2$.
$y = \frac{4}{x - 1} + 2$

c) a square root function whose graph has a vertex at V(−4, −2) and passes through the point A(5, 4). $y = 2\sqrt{x + 4} - 2$

3.1 Function

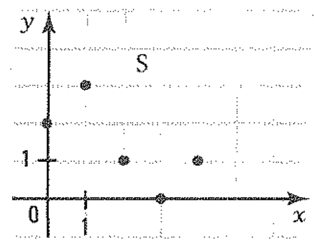
ACTIVITY 1 Recognizing a function

a) Consider the mapping diagram of the relation R represented on the right.



1. What is the source set? *A = {2, 3, 4, 5}*
2. What is the target set? *B = {4, 5, 6, 7, 8}*
3. Complete: An element x from set A is in relation with an element y from set B if x is *a divisor* of y.
4. Is there an element from the source set that is in relation with more than one element from the target set? *Yes*
5. Is this relation a function? Justify your answer.
   No, 2 is in relation with three elements and 4 with two elements.

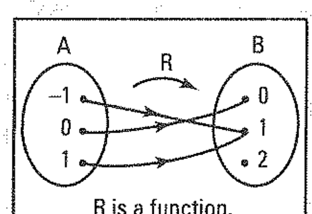
b) Consider the Cartesian graph of the relation S represented on the right. The point (1, 3) means that the element 1 from the source set is in relation with the element 3 from the target set.



1. What is the image of 4? *1*
2. What is the antecedent of 2? *0*
3. Is there an element from the source set that is in relation with more than one element from the target set? *No*
4. Is this relation a function? Justify your answer.
   Yes, since each element from the source set is in relation to at most one element from the target set.

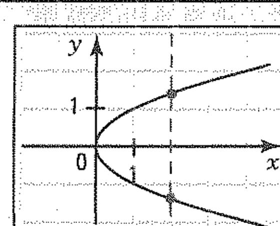
DEFINITION OF A FUNCTION

- A relation given by a source set A to a target set B is a **function** if each element from A is associated with at most one element from B.
- **Mapping diagram**
  Given the mapping diagram of a relation, this relation is a function if, from each element of the source set, at most one arrow is drawn.



R is a function.

- **Cartesian graph**
  Given the Cartesian graph of a relation, this relation is a function if any vertical line intersects the graph of this relation in at most one point.



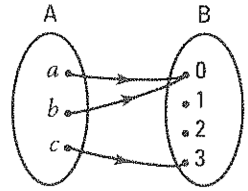
R is not a function.

- **Set of ordered pairs**
  Given a relation's set of ordered pairs, this relation is a function if the first coordinate of each pair verifying the relation appears only once.

> $G_R = \{(\underline{a}, 0), (b, 1), (\underline{a}, 2)\}$
>
> R is not a function.

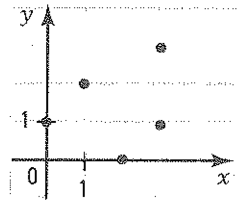
1. In each of the following cases, indicate if the relation is a function.

a)



Yes

b)

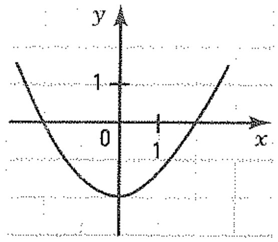


No

c) $G = \{(0, 0), (1, -1), (1, 1)\}$

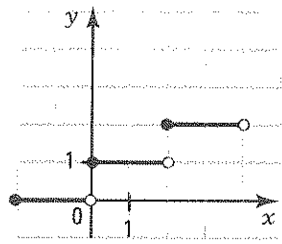
No

d)



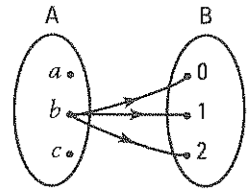
Yes

e)



Yes

f)

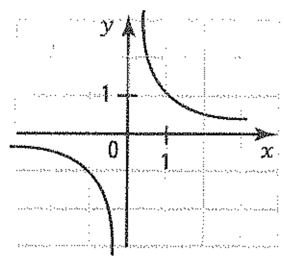


No

g) $G = \{(4, 3), (5, 3), (6, 3)\}$

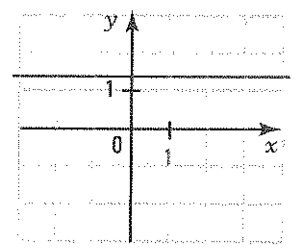
Yes

h)



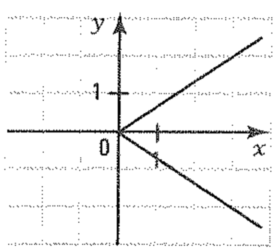
Yes

i)



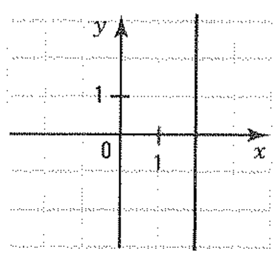
Yes

j)



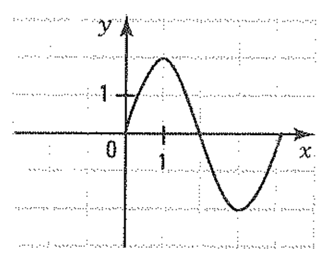
No

k)



No

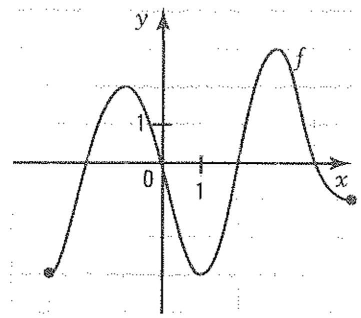
l)



Yes

ACTIVITY 2 Properties of functions

Consider the function f represented on the right.



a) What is the domain of f? ***[–3, 5]***

b) What is the range of f? ***[–3, 3]***

c) What are the zeros of f? ***–2, 0, 2 and 4***

d) What is the initial value of f? ***0***

e) Over what interval is the function f

1. positive? ***[–2, 0] ∪ [2, 4]*** 2. negative? ***[–3, –2] ∪ [0, 2] ∪ [4, 5]***

f) Over what interval is the function f

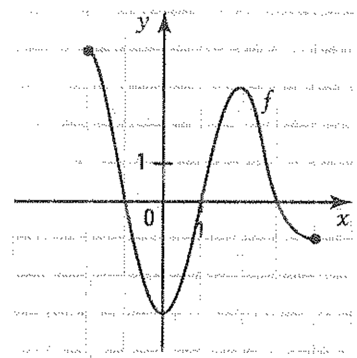
1. increasing? ***[–3, –1] ∪ [1, 3]*** 2. decreasing? ***[–1, 1] ∪ [3, 5]***

g) What is, for function f, its

1. absolute maximum? ***3*** 2. absolute minimum? ***–3***

PROPERTIES OF FUNCTIONS

Consider the function f represented on the right.



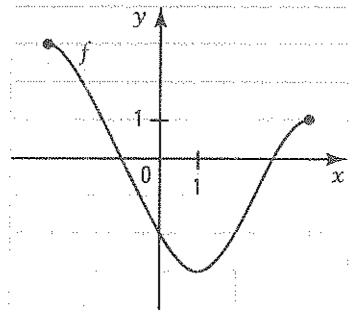
- The **domain** of a function f is the subset of the elements of the source set which have an image in f.
  dom $f = [-2, 4]$
- The **range** of a function f is the subset of the elements of the target set which are images by f.
  ran $f = [-3, 4]$
- The **zeros** of the function f are the values of x for which the function is equal to zero. The zeros of f are: –1, 1 and 3.
- The **initial value** of the function f is the value of y when $x = 0$. The initial value of f is –3.
- Studying the **sign** of a function consists of finding the values of x for which the function is positive or those for which the function is negative.

  $f(x) \geq 0$ if $x \in [-2, -1] \cup [1, 3]$.
  $f(x) \leq 0$ if $x \in [-1, 1] \cup [3, 4]$.
- Studying the **variation** of a function consists of finding the values of x for which the function is increasing or those for which the function is decreasing..

  f is increasing if $x \in [0, 2]$.
  f is decreasing if $x \in [-2, 0] \cup [2, 4]$.
- The **absolute maximum** (or **minimum**) of a function is the highest image (or the lowest image) when it exists.

  max $f = 4$, min $f = -3$

2. Consider the function f represented on the right. Determine



a) 1. dom f = *[–3, 4]* 2. ran f = *[–3, 3]*

b) 1. the zeros of f: *–1 and 3*

2. the initial value: *–2*

c) the values of x for which the function f is

1. positive: *[–3, –1] ∪ [3, 4]* 2. negative: *[–1, 3]*

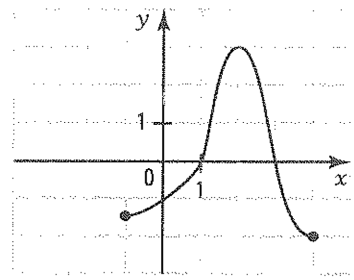
d) the values of x for which the function f is

1. increasing: *[1, 4]* 2. decreasing: *[–3, 1]*

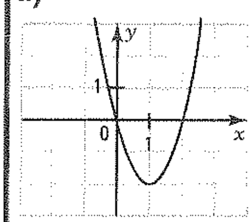
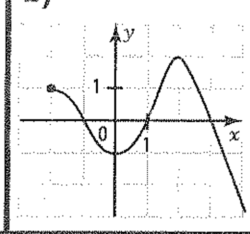
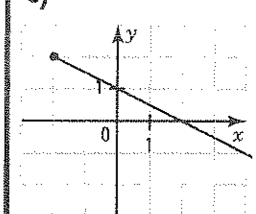
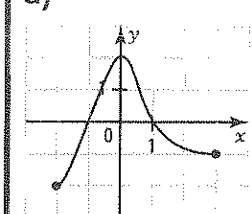
e) 1. the maximum of f: *3* 2. the minimum of f: *–3*

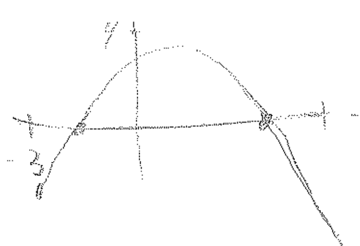
3. Draw the graph of a function satisfying the following conditions.

1. dom $f = [-1, 4]$.
2. ran $f = [-2, 3]$.
3. The zeros of f are: 1 and 3.
4. The initial value is –1.
5. The function is negative when $x \in [-1, 1] \cup [3, 4]$.
6. The function is increasing when $x \in [-1, 2]$ and decreasing when $x \in [2, 4]$.
7. max $f = 3$ and min $f = -2$.



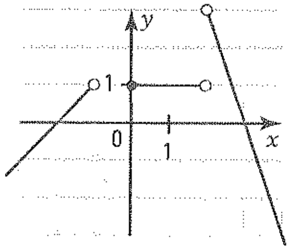
4. Study the following functions by completing the table below.

	a) 	b) 	c) 	d) 
domain	ℝ	*[–2, +∞[*	*[–2, +∞[*	*[–2, 3]*
range	*[–2, +∞[*	*]–∞, 2]*	*]–∞, 2]*	*[–2, 2]*
zeros	*0 and 2*	*–1, 1 and 3*	*2*	*–1 and 1*
initial value	*0*	*–1*	*1*	*2*
$f(x) \geq 0$ if $x \in$	*]–∞, 0] ∪ [2, +∞[*	*[–2, –1] ∪ [1, 3]*	*[–2, 2]*	*[–1, 1]*
$f(x) \leq 0$ if $x \in$	*[0, 2]*	*[–1, 1] ∪ [3, +∞[*	*[2, +∞[*	*[–2, –1] ∪ [1, 3]*
$f \nearrow$ if $x \in$	*[1, +∞[*	*[0, 2]*	*never*	*[–2, 0]*
$f \searrow$ if $x \in$	*]–∞, 1]*	*[–2, 0] ∪ [2, +∞[*	*[–2, +∞[*	*[0, 3]*
extrema	*min f = –2*	*max f = 2*	*max f = 2*	*max f = 2,* *min f = –2*



5. Determine the domain and range of the following functions.

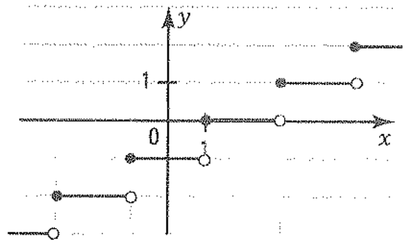
a)



dom =]−∞, −1[∪ [0, 2[∪]2, +∞[

ran =]−∞, 2[3 [

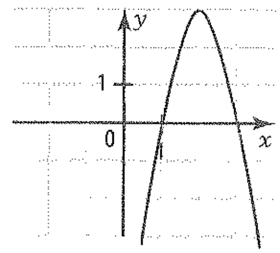
b)



dom = ℝ

ran = ℤ

c)

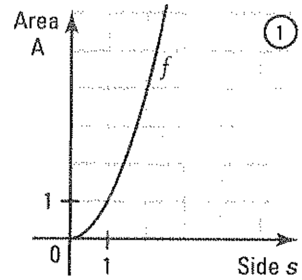


dom = ℝ

ran =]−∞, 3]

ACTIVITY 3 Inverse of a function

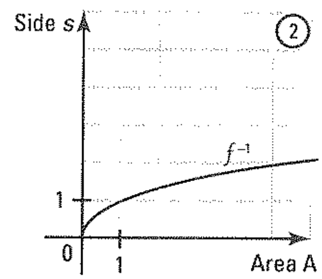
Let s represent the side of a square and A represent its area.



a) 1. What is the rule of the function f that associates, to the square's side s, its area? *$A = s^2$*

2. Complete the table of values below and represent the function f in the Cartesian plane ①.

Side s	0	0.5	1	1.5	2
Area A	*0*	*0.25*	*1*	*2.25*	*4*



b) 1. What is the rule of the inverse f^{-1} that associates, to the square's area A, its side length s?

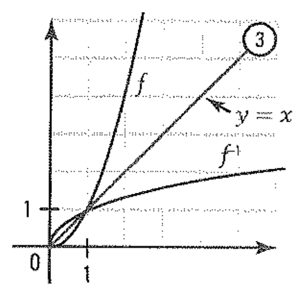
$s = \sqrt{A}$

2. Complete the table of values below and represent the function f^{-1} in the Cartesian plane ②.

Area A	0	0.25	1	2.25	4
Side s	*0*	*0.5*	*1*	*1.5*	*2*

3. Explain why the inverse f^{-1} is a function.

Any vertical line only intersects the curve at a maximum of one point.

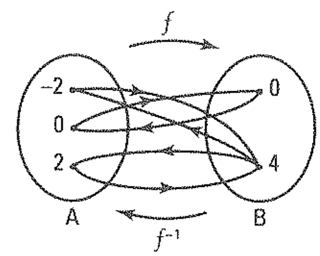


c) 1. Reproduce the two graphs in the same Cartesian plane ③ where the axes are not labeled.

2. Verify that the graphs of f and f^{-1} are symmetrical about the bisector of the 1st quadrant.

ACTIVITY 4 Functions whose inverse is not a function

a) Consider the sets A and B on the right, and the function f of A toward B with the rule $f(x) = x^2$.



1. Use a mapping diagram to represent function f.
2. Deduce the mapping diagram of the inverse f^{-1}.
3. Explain why f^{-1} is not a function.

4 is in relation with two elements −2 and 2 by f^{-1}. Therefore, f^{-1} is not a function.

b) Consider the table of values on the right of a function f.

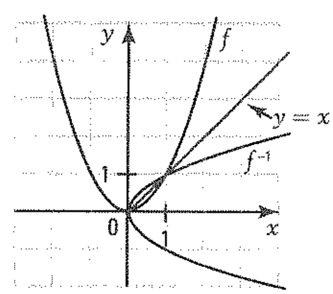
x	−2	−1	0	1	2
$f(x)$	2	1	0	1	2

1. Deduce a table of values for f^{-1}.
2. Explain why f^{-1} is not a function.

x	2	1	0	1	2
$f^{-1}(x)$	−2	−1	0	1	2

1 is in relation with two elements −1 and 1.

c) The function f on the right has the rule $f(x) = x^2$.



1. Deduce, by symmetry about the bisector of the 1st quadrant, the graph of the inverse f^{-1}.
2. Explain why the inverse f^{-1} is not a function.

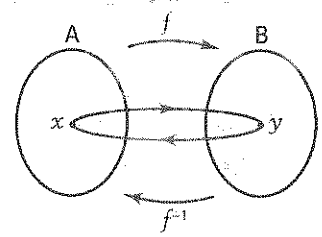
There is a vertical line that intersects the graph of f^{-1} at 2 points.

3. True or false?

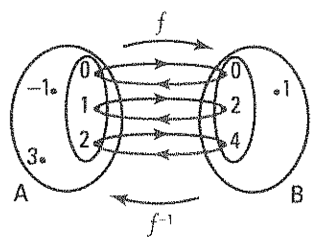
The inverse of f is not a function when a horizontal line can be drawn to intersect the graph of f at more than one point. *True*

INVERSE OF A FUNCTION

- If f is the function of a source set A toward a target set B, the inverse of f, written f^{-1}, has the source set B and the target set A.
- The inverse of a function is not necessarily a function.

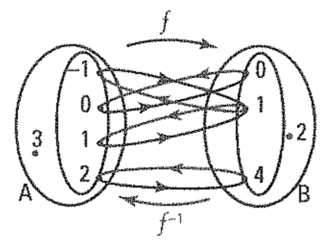


Ex.: f: A → B
$x \mapsto y = 2x$



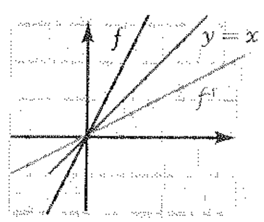
f^{-1} is a function.
$\text{dom} f = \text{ran} f^{-1} = \{0, 1, 2\}$
$\text{ran} f = \text{dom} f^{-1} = \{0, 2, 4\}$

Ex.: f: A → B
$x \mapsto y = x^2$

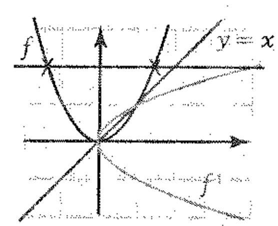


f^{-1} is not a function.
$\text{dom} f = \text{ran} f^{-1} = \{-1, 0, 1,2\}$
$\text{ran} f = \text{dom} f^{-1} = \{0, 1, 4\}$

- For any function f, we have: $\text{dom} f = \text{ran} f^{-1}$ and $\text{ran} f = \text{dom} f^{-1}$
- The Cartesian graphs of a function and its inverse are **symmetrical** about the line with the equation $y = x$.



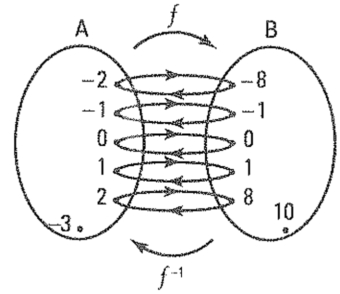
f^{-1} is a function.



f^{-1} is not a function.

- The inverse of a function f is not a function when a **horizontal line** can be drawn to intersect the graph of f **at more than one point**.

6. Consider the mapping diagram of a function f.



a) Deduce the mapping diagram of f^{-1}.

b) Explain why f^{-1} is a function.

There is at most one arrow from each element of the source set B.

c) Determine

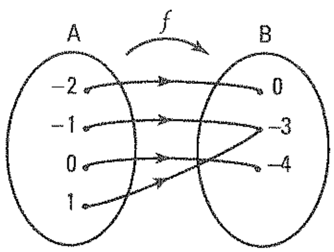
1. dom$f$. *{-2, -1, 0, 1, 2}*
2. ran f. *{-8,-1, 0, 1, 8}*
3. domf^{-1}. *{-8,-1, 0, 1, 8}*
4. ran f^{-1}. *{-2, -1, 0, 1, 2}*

d) Verify that

1. dom$f$ = ranf^{-1}.
2. ran$f$ = domf^{-1}.

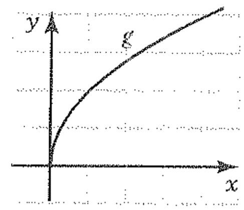
7. Indicate which of the following functions have an inverse that is also a function.

a)



No

b)

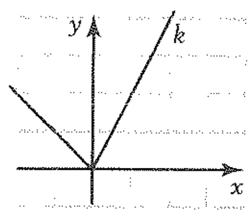


Yes

c) $h = \{(-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4)\}$

Yes

d)

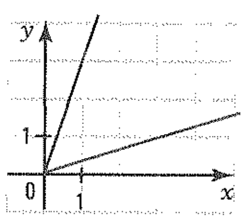


No

8. For each of the following functions,

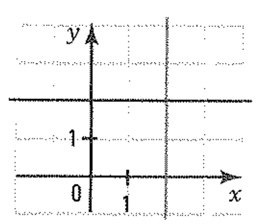
1. deduce the graph of the inverse.
2. indicate if the inverse is a function.

a)



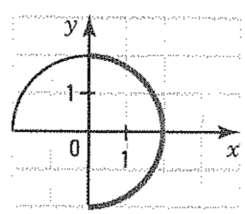
Yes

b)



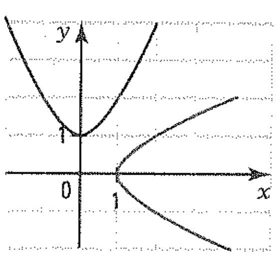
No

c)



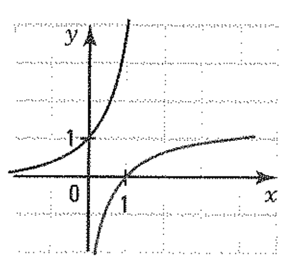
No

d)



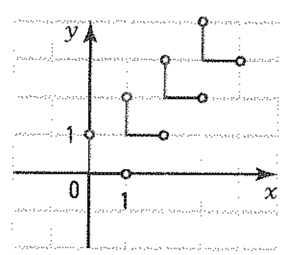
No

e)



Yes

f)



No

Activity 5 Rule of the inverse

A salesman in a store receives a weekly base salary of \$250 and a sales commission of \$10 per item sold for the week.

a) Let a represent the number of items sold for the week, and s represent the total weekly salary. Determine the rule of

1. the function f which gives the total salary s as a function of the number of items sold a. $s = 250 + 10a$

2. the function f^{-1} which associates, to a given salary s, the number of items sold a. $a = \frac{s - 250}{10}$

b) Complete the table of values on the right for the functions f and f^{-1}.

a	0	5	10	15	20
s	250	300	350	400	450

(f: $a \to s$; f^{-1}: $s \to a$)

RULE OF THE INVERSE

Given the function f with the rule: $y = 2x + 6$. To determine the rule of the inverse f^{-1},

1. we isolate x in the rule of f.

$$y = 2x + 6$$
$$2x = y - 6$$
$$x = \frac{1}{2}y - 3$$

2. we switch the letters x and y.

$$y = \frac{1}{2}x - 3$$

f^{-1} therefore has the rule: $y = \frac{1}{2}x - 3$.

We interchange the letters x and y to respect the convention of function notation which assigns x as elements of the source set and y as elements of the target set.

9. For each of the following rules of functions, find the rule of its inverse.

a) $y = 5x$ — $y = \frac{x}{5}$

b) $y = 3x - 6$ — $y = \frac{x}{3} + 2$

c) $y = -2x + 10$ — $y = \frac{-x}{2} + 5$

d) $y = 0.1x + 100$ — $y = 10x - 1000$

e) $y = \frac{2}{3}x - 6$ — $y = \frac{3}{2}x + 9$

f) $y = -\frac{3}{4}x + 12$ — $y = \frac{-4}{3}x + 16$

10. A capital of \$1000 is invested on January 1st, 2009 at an annual interest rate of 10%. Find the rule which associates

a) a given number of elapsed years t since the beginning, to the accumulated capital C.
$C = 1000 + 100t$

b) a given accumulated capital C, to the number of elapsed years t. $t = 0.01C - 10$

11. A car's gas tank initially contains 60 litres of gas. This car consumes on average 12 litres/100 km. Find the rule of the function which associates,

a) a given distance traveled d (in km) to the quantity q of gas remaining in the tank.
$q = -0.12d + 60$

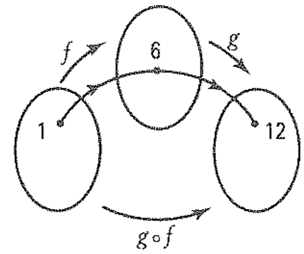
b) a given quantity q of gas remaining in the tank, to the distance traveled d (in km).
$d = -\frac{25}{3}q + 500$

ACTIVITY 6 Composition of functions

Consider the function f defined by $f(x) = x + 5$ and the function g defined by the rule $g(x) = 2x$.

a) Determine

1. $f(1)$ **6**
2. $g(f(1))$ **12**

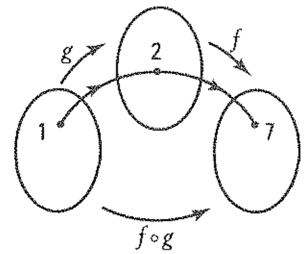


b) The composition of f by g, written $g \circ f$ is defined by $g \circ f(x) = g(f(x))$.

1. Calculate $g \circ f(1)$ **12**
2. Determine the rule of $g \circ f$. $g \circ f(x) = g(f(x)) = g(x + 5) = 2x + 10$

c) Determine

1. $g(1)$ **2**
2. $f(g(1))$ **7**



d) The composition of g by f, written $f \circ g$, is defined by $f \circ g(x) = f(g(x))$.

1. Calculate $f \circ g(1)$ **7**
2. Determine the rule of $f \circ g$. $f \circ g(x) = f(g(x)) = f(2x) = 2x + 5$

e) Compare the rules of $g \circ f$ and $f \circ g$.

$g \circ f(x) \neq f \circ g(x)$

COMPOSITION OF FUNCTIONS

- Given two functions f and g,
  - the composition of f by g, written $g \circ f$, is defined by the rule:

$$g \circ f(x) = g(f(x))$$

  - the composition of g by f, written $f \circ g$, is defined by the rule:

$$f \circ g(x) = f(g(x))$$

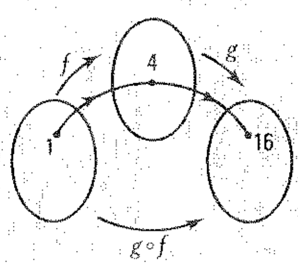
Ex.: Given $f(x) = x + 3$ and $g(x) = x^2$, we have:

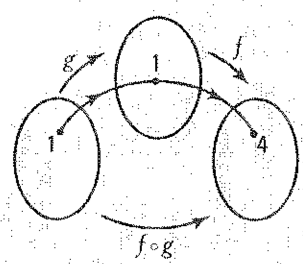
$g \circ f(1) = g(f(1)) = g(4) = 16$;

$g \circ f(x) = g(x + 3) = (x + 3)^2$;

$f \circ g(1) = f(g(1)) = f(1) = 4$

$f \circ g(x) = f(x^2) = x^2 + 3$





Note that, in general, $g \circ f(x) \neq f \circ g(x)$.

12. Consider the functions $f(x) = 3x - 5$ and $g(x) = -2x + 8$. Determine

a) $g \circ f(2) =$ **6**
b) $f \circ g(-1) =$ **25**
c) $f \circ g(4) =$ **-5**
d) $g \circ f(0) =$ **18**
e) $g \circ g(7) =$ **20**
f) $f \circ g(-5) =$ **49**

13. Consider the functions $f(x) = -2x + 5$ and $g(x) = 4x - 3$. Determine the rules of the following functions.

a) $f \circ g(x) =$ $f(g(x)) = f(4x - 3) = -2(4x - 3) + 5 = -8x + 11$

b) $g \circ f(x) =$ $g(f(x)) = g(-2x + 5) = 4(-2x + 5) - 3 = -8x + 17$

c) $f \circ f(x) =$ $f(f(x)) = f(-2x + 5) = -2(-2x + 5) + 5 = 4x - 5$

d) $g \circ g(x) =$ $g(g(x)) = g(4x - 3) = 4(4x - 3) - 3 = 16x - 15$

14. Consider the functions $f(x) = 2x + 3$ and $g(x) = 3x - 2$.

a) Determine the rule of

1. $g \circ f$. $g \circ f(x) = 6x + 7$ 2. $f \circ g$. $f \circ g(x) = 6x - 1$

b) Verify that $g \circ f(x) \neq f \circ g(x)$.

15. Consider $f(x) = x + 5$ and $g(x) = x - 2$. Verify that, $g \circ f(x) = f \circ g(x)$. slope = 1

$g \circ f(x) = x + 3,\ f \circ g(x) = x + 3$

16. Consider the function $f(x) = 2x + 8$.

a) Determine the rule of the inverse f^{-1}. $f^{-1}(x) = \frac{1}{2}x - 4$

b) 1. Determine the rule of the composite $f^{-1} \circ f$.

$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(2x + 8) = \frac{1}{2}(2x + 8) - 4 = x$

2. Determine the rule of the composite $f \circ f^{-1}$.

$f \circ f^{-1}(x) = f(f^{-1}(x)) = f\left(\frac{1}{2}x - 4\right) = 2\left(\frac{1}{2}x - 4\right) + 8 = x$

3. Verify that $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$.

c) Repeat this exercise with the function $f(x) = -5x + 10$.

$f^{-1}(x) = -\frac{1}{5}x + 2;\ f^{-1} \circ f(x) = x;\ f \circ f^{-1}(x) = x$

17. Consider the functions $f(x) = x + 5$ and $f(x) = 3x + 4$.

a) Determine the rule of the functions f^{-1} and g^{-1}.

$f^{-1}(x) = x - 5$ $g^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$

b) Determine

1. $f \circ f^{-1}(x) =$ $f(f^{-1}(x)) = f(x - 5) = x - 5 + 5 = x$

2. $g \circ g^{-1}(x) =$ $g(g^{-1}(x)) = g\left(\frac{1}{3}x - \frac{4}{3}\right) = 3\left(\frac{1}{3}x - \frac{4}{3}\right) + 4 = x$

3. $f \circ g(x) =$ $f(g(x)) = f(3x + 4) = 3x + 4 + 5 = 3x + 9$

4. $g \circ f(x) =$ $g(f(x)) = g(x + 5) = 3(x + 5) + 4 = 3x + 19$

c) Determine

1. $(f \circ g)^{-1}(x) =$ $\frac{1}{3}x - 3$ 2. $(g \circ f)^{-1}(x) =$ $\frac{1}{3}x - \frac{19}{3}$

3. $g^{-1} \circ f^{-1}(x) =$ $\frac{1}{3}x - 3$ 4. $f^{-1} \circ g^{-1}(x) =$ $\frac{1}{3}x - \frac{19}{3}$

d) What can you deduce? $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$ and $(g \circ f)^{-1}(x) = f^{-1} \circ g^{-1}(x)$

18. Consider the functions $f(x) = x^2 + 4x - 5$ and $g(x) = 2x - 1$.

a) Determine the rule of the composite $f \circ g$.

$f \circ g(x) = f(g(x)) = f(2x - 1) = (2x - 1)^2 + 4(2x - 1) - 5 = 4x^2 + 4x - 8$

b) Determine $f \circ g(2)$ in two different ways:

1. by finding $f(g(2)) =$ $f(3) = 16$
2. by using the rule found in a). $4(2)^2 + 4(2) - 8 = 16$

19. In Quebec, every purchase is taxable. The goods and services tax (GST) is 5 %. The Quebec sales tax (QST) is 7.5 %.

Let f be the function which associates a given purchase amount x to the amount y including GST.

Let g be the function which associates a given purchase amount x to the amount y including QST.

a) Determine the rule of the function

1. f: $y = 1.05x$ 2. g: $y = 1.075x$

b) 1. Determine the rule of the function $g \circ f$. $g \circ f(x) = 1.12875x$ $1.075(1.05x)$

2. Determine the rule of the function $f \circ g$. $f \circ g(x) = 1.12875x$

c) Compare the rules of the functions $g \circ f$ and $f \circ g$. What can you conclude?

The rules are equal. To calculate the final price of a product, it doesn't matter if you apply the GST first and then the QST, or the QST first and then the GST.

d) 1. What is the final price of a product with a \$39.80 price tag? **\$44.92**

2. What is the initial price tag of a product if the final cost paid is \$56.44? **\$50**

20. The weekly salary of a sporting goods store salesman includes a base salary of \$300 per week and a \$40 bonus for every item sold.

During the holidays, the owner of the store decides to give each employee a 4% bonus on their weekly salary.

Let f be the function which gives the regular weekly salary y as a function of the number of items sold x.

Let g be the function which gives the bonus holiday weekly salary y as a function of the regular weekly salary x.

a) Determine the rule of the function $1.04(40x + 300)$

1. f: $y = 40x + 300$ 2. g: $y = 1.04x$ 3. $g \circ f$: $y = 41.6x + 312$

b) What will an employee's salary be, during the holidays, if he sells 4 items during the week? **\$478.40**

c) How many items did an employee sell if he receives a weekly salary of \$561.60 during the holidays? *6 items*

$561.6 = 41.6x + 312$

Activity 7 Operations between functions

Consider the functions $f(x) = x^2 - 9$ and $g(x) = x + 3$. Determine

a) $f(x) + g(x) =$ $x^2 + x - 6$

b) $f(x) - g(x) =$ $x^2 - x - 12$

c) $f(x) \times g(x) =$ $x^3 + 3x^2 - 9x - 27$

d) $\frac{f(x)}{g(x)} =$ $x - 3$

OPERATIONS BETWEEN FUNCTIONS

Given two real functions f and g, we have:

$(f + g)(x) = f(x) + g(x)$

$(f - g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \times g(x)$

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$

Ex.: Given $f(x) = x^2 + 2x - 15$ and $g(x) = 2x - 6$, we have:

$(f + g)(x) = f(x) + g(x) = (x^2 + 2x - 15) + (2x - 6) = x^2 + 4x - 21.$

$(f - g)(x) = f(x) - g(x) = (x^2 + 2x - 15) - (2x - 6) = x^2 - 9.$

$(f \cdot g)(x) = f(x) \times g(x) = (x^2 + 2x - 15)(2x - 6) = 2x^3 - 2x^2 - 42x + 90.$

$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2 + 2x - 15}{2x - 6} = \frac{(x - 3)(x + 5)}{2(x - 3)} = \frac{x + 5}{2}.$

21. Consider the four functions f, g, h, and i. Let $f(x) = x^2 + x - 6$, $g(x) = 2x - 4$, $h(x) = x^2 - 9$ and $i(x) = 3x^2 - 12$.

a) $(f + g + h)(x) =$ $2x^2 + 3x - 19$

b) $(f - g + h)(x) =$ $2x^2 - x - 11$

c) $(f \cdot g)(x) =$ $2x^3 - 2x^2 - 16x + 24$

d) $(g \cdot h)(x) =$ $2x^3 - 4x^2 - 18x + 36$

e) $(f - h - i)(x) =$ $-3x^2 + x + 15$

f) $\left(\frac{f}{g}\right)(x) =$ $\frac{x + 3}{2}$ $(x \neq -3)$

g) $\left(\frac{f \cdot g}{i}\right)(x) =$ $\frac{2(x + 3)(x - 2)}{3(x + 2)}$ $(x \neq 2)$

h) $\left(\frac{g \cdot h}{f}\right)(x) =$ $2(x - 3)$ $(x \neq -3$ and $x \neq 2)$

22. The condominium association of a building establishes the following fees to be charged to each of its condo owners.

- Monthly condo fees: \$225
- Monthly fees for renovations: \$80
- Municipal taxes paid at the beginning of the year: \$1500

a) Determine the rule of the function f which gives the cost y of condo fees as a function of the number x of months. $y = 225x$

b) Determine the rule of the function g which gives the total cost y of renovation fees and municipal taxes as a function of the number x of months. $y = 80x + 1500$

c) Determine the rule of the function $f + g$ and interpret this rule. $y = 305x + 1500$

f + g gives the total fees charged to a condo owner as a function of the number x of months.

d) What is the total amount of fees paid by a condo owner after 8 months of occupancy?

\$3940

3.2 Polynomial functions

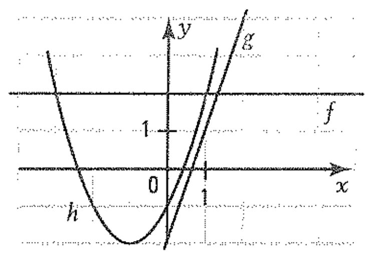
ACTIVITY 1 Polynomial functions

a) Among the following functions, indicate which ones are polynomial functions. If it is a polynomial function, indicate its degree.

1. $P(x) = -5x + 8$ ***Yes, 1st degree***
2. $P(x) = -4x^2 - 5x$ ***Yes, 2nd degree***
3. $P(x) = \frac{5}{x} + 3$ ***No***
4. $P(x) = -3$ ***Yes, degree 0***
5. $P(x) = \sqrt{x} - 7$ ***No***
6. $P(x) = x^3 + 4x^2 - 5x + 3$ ***Yes, 3rd degree***

b) Represent the following polynomial functions in the Cartesian plane on the right.

1. $f(x) = 2$ 2. $g(x) = 3x - 2$ 3. $h(x) = x^2 + 2x - 1$



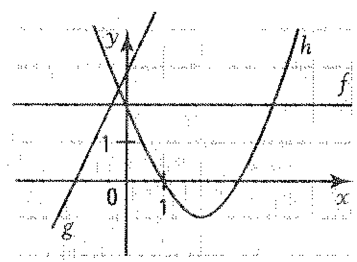
POLYNOMIAL FUNCTIONS

- A **polynomial function** is any function with a polynomial for a rule.

  Ex.: $f(x) = 2$ is a zero degree polynomial function.

  $g(x) = 2x + 3$ is a 1st degree polynomial function.

  $h(x) = x^2 - 4x + 3$ is a 2nd degree polynomial function.

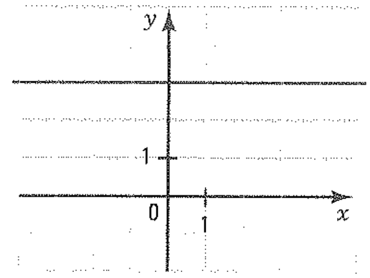


- The following table classifies polynomial functions according to their degree.

Degree	Basic polynomial function	Transformed polynomial function	Name
0	$f(x) = 1$	$f(x) = b$ where $b \in \mathbb{R}$	constant function
1	$f(x) = x$	$f(x) = ax$ where $a \in \mathbb{R}^*$	direct variation linear function
		$f(x) = ax + b$ where $a, b \in \mathbb{R}^*$	partial variation linear function
2	$f(x) = x^2$	$f(x) = ax^2 + bx + c$ where $a \in \mathbb{R}^*$	quadratic function
3	$f(x) = x^3$	$f(x) = ax^3 + bx^2 + cx + d$ where $a \in \mathbb{R}^*$	cubic function

ACTIVITY 2 Study of a constant function

Consider the function f given by the rule $y = 3$.



a) Represent this function in the Cartesian plane.

b) Determine

1. dom $f =$ $\mathbb{R}$ 2. ran $f =$ *{3}*
3. the zeros of f if they exist. *No zeros*
4. the y-intercept. *3*
5. the sign of f *f(x) ≥ 0 over ℝ*
6. the variation of f *f is a constant function* 7. the extrema of f *max f = min f = 3*

c) What is the rate of change between two random points on the graph of f? *It is zero.*

CONSTANT FUNCTIONS

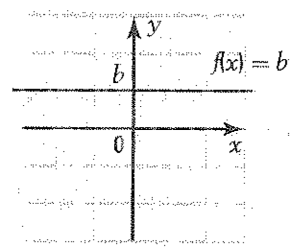
- A **constant function** is a zero degree polynomial function. It is described by a rule of the form:

$$\boxed{f(x) = b}, b \in \mathbb{R}$$

- The Cartesian graph of a constant function is a horizontal line with the equation $y = b$.

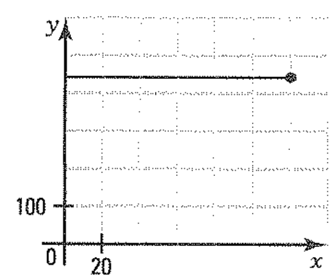
Study of a constant function

- dom $f = \mathbb{R}$
- ran $f = \{b\}$
- The constant function has no zero unless $b = 0$.
- $f(x) > 0$ over $\mathbb{R}$ if $b > 0$
  $f(x) < 0$ over $\mathbb{R}$ if $b < 0$
- max $f =$ min $f = b$



- The rate of change of any constant function is zero.
- A **zero function** is a constant function described by the rule $f(x) = 0$. Its Cartesian graph is represented by the x-axis.

1. A ski resort is open 120 days during the ski season. The cost of a season pass is $450. Consider the function f which gives the total cost y as a function of the number x of days of skiing.



a) How much does it cost to ski for 12 days? *$450*

b) What is the rule of function f? *y = 450*

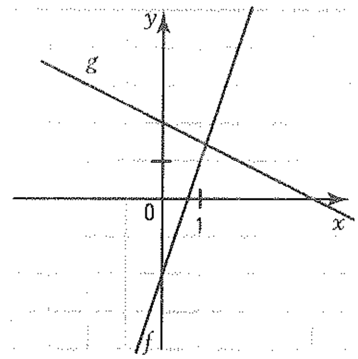
c) Represent function f in the Cartesian plane.

d) Determine

1. dom $f =$ *[0, 120]* 2. ran $f =$ *{450}*

ACTIVITY 3 Study of a linear function

Consider the functions $f(x) = 3x - 2$ and $g(x) = -\frac{1}{2}x + 2$.



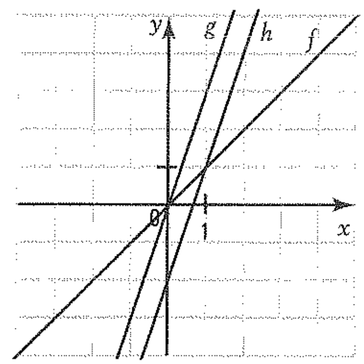
a) Represent the functions f and g in the Cartesian plane on the right.

b) Study the functions f and g and complete the following table.

	Function f	Function g
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$\mathbb{R}$	$\mathbb{R}$
Zero	$\frac{2}{3}$	4
Initial value	-2	2
Sign	$f(x) \geqslant 0$ if $x \in \left[\frac{2}{3}, +\infty\right[$ $f(x) \leqslant 0$ if $x \in \left]-\infty, \frac{2}{3}\right]$	$f(x) \geqslant 0$ if $x \in]-\infty, 4]$ $f(x) \leqslant 0$ if $x \in [4, +\infty[$
Variation	*f is increasing over* $\mathbb{R}$	*g is decreasing over* $\mathbb{R}$

ACTIVITY 4 Transformations of the basic linear function

The basic 1st degree linear function $f(x) = x$ is represented on the right.



a) 1. Draw the image of function f by the vertical scale change $(x, y) \rightarrow (x, 3y)$ to obtain the graph of function g.

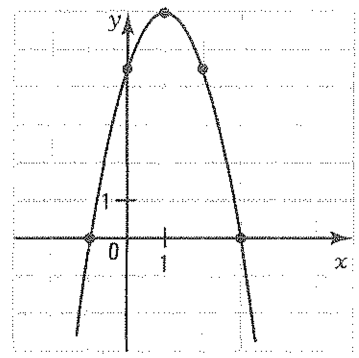
2. What is the rule of the function g? $g(x) = 3x$

b) 1. Draw the image of function g by the vertical translation $(x, y) \rightarrow (x, y - 2)$ to obtain the graph of function h.

2. What is the rule of the function h? $h(x) = 3x - 2$

ACTIVITY 6 Study of a quadratic function (standard form)

Consider the function f given by the rule $y = -1.5(x - 1)^2 + 6$.



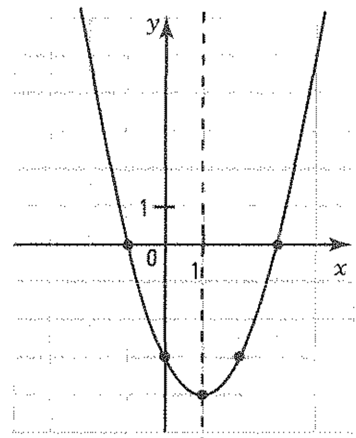
a) Represent this function in the Cartesian plane.

b) Determine

1. dom $f =$ ℝ
2. ran $f =$]−∞, 6]
3. the zeros of f. **−1 and 3**
4. the initial value of f. **4.5**
5. the sign of f. pos. $f(x) \geq 0$ over $x \in [-1, 3]$;
   neg. $f(x) \leq 0$ over $x \in]-\infty, -1] \cup [3, +\infty[$
6. the variation of f. $f \nearrow$ over $x \in]-\infty, 1]$; inc.
   $f \searrow$ over $x \in [1, +\infty[$ dec.
7. the extrema of f. **max $f = 6$**

ACTIVITY 7 Study of a quadratic function (general form)

Consider the function f given by the rule $y = x^2 - 2x - 3$.



a) Is the parabola representing f open upward or downward?
Upward, $a > 0$.

b) What are the coordinates of the vertex? **V(1, −4)**

c) Determine the zeros of the function f. $x_1 = -1$ and $x_2 = 3$

d) What is the initial value of f? **−3**

e) What is the equation of the axis of symmetry? $x = 1$

f) Represent this function in the Cartesian plane.

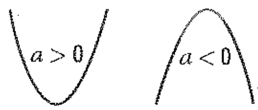
g) Determine:

1. dom $f =$ ℝ
2. ran $f =$ [−4, +∞[
3. the sign of f. $f(x) \geq 0$ over $x \in]-\infty, -1] \cup [3, +\infty[$; $f(x) \leq 0$ over $x \in [-1, 3]$
4. the variation of f. $f \searrow$ over $x \in]-\infty, 1]$; $f \nearrow$ over $x \in [1, +\infty[$
5. the extrema of f. **min $f = -4$**

QUADRATIC FUNCTION

Standard form

$f(x) = a(x - h)^2 + k$



General form

$f(x) = ax^2 + bx + c$

- Vertex: $V(h, k)$
- Axis of symmetry: $x = h$
- Zeros: $h \pm \sqrt{-\frac{k}{a}}$

- Vertex: $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$
- Axis of symmetry: $x = -\frac{b}{2a}$
- Zeros: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- y-intercept: c

Factored form

$f(x) = a(x - x_1)(x - x_2)$

- Zeros: x_1 and x_2
- Axis of symmetry: $x = \frac{x_1 + x_2}{2}$

4. Determine the domain and range of the following functions.

a) $f(x) = -3(x - 2)^2 + 5$

dom f = ℝ

ran f =]−∞, 5]

b) $f(x) = 2x^2 + 4x - 9$

dom f = ℝ

ran f = [−11, +∞[

5. Determine the zeros of the function $f(x) = -3(x + 1)^2 + 12$. ***$x_1 = -3$ and $x_2 = 1$***

6. Determine the y-intercept of $f(x) = -\frac{1}{2}(x + 4)^2 + 9$. ***y = 1***

7. Determine over what interval the function $f(x) = 2x^2 - 5x - 3$ is positive. Find zeros

$f(x) \geq 0$ over $\left]-\infty, -\frac{1}{2}\right] \cup [3, +\infty[$ $(2x + 1)(x - 3)$ $-\frac{1}{2}, 3$

✓

8. Determine over what interval the function $f(x) = 3x^2 + 6x - 5$ is increasing. ***[−1, +∞[***

9. Determine the extrema of the function $f(x) = -2x^2 + 12x - 7$. ***max f = 11***

10. What is the axis of symmetry of the function $f(x) = -\frac{1}{4}x^2 + 3x + 1$? ***x = 6***

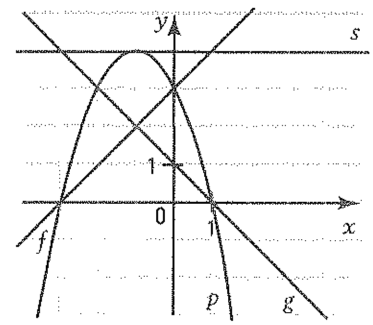
11. Determine the values of x for which the function $f(x) = -3(x + 4)^2 + 5$ is equal to −7.

x = −6 or x = −2

12. Find the rule of the quadratic function represented by a parabola with a vertex at V(−1, 5) and passing through the point P(1, 3).

$y = -\frac{1}{2}(x + 1)^2 + 5$

13. Consider the functions $f(x) = x + 3$ and $g(x) = -x + 1$ represented on the right.



a) Represent the function s given that $s(x) = f(x) + g(x)$.

$s(x) = 4$

b) Represent the function p given that $p(x) = f(x) \cdot g(x)$.

$p(x) = -x^2 - 2x + 3$

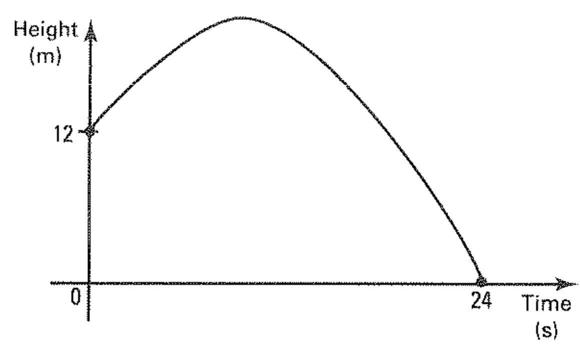
14. A stone is thrown upward from the top of a seaside cliff. The function which gives the stone's height h (in m) above sea level as a function of time t (in sec) since it was thrown has the rule: $h = -t^2 + 12t + 160$.
Find the interval of time over which the height of the stone is at least 180 m above sea level.

Between the instants $t = 2$ and $t = 10$ seconds after it was thrown.

15. The height h, in metres, of a diver relative to the water level is described by the rule $h = \frac{1}{2}t^2 - 6t + 10$ where t represents the elapsed time, in seconds, since the start of the dive. How long did the diver remain underwater?

During 8 seconds.

16. A projectile is thrown upward from a height of 12 m. After 10 seconds, it reaches its maximum height and after 24 seconds, it hits the ground. Knowing that its trajectory follows the rule of a quadratic function, find the elapsed time between the moment it reaches a height of 6.5 m, on its descent, and the time when it hits the ground.



$y = -\frac{1}{8}(x + 4)(x - 24)$.

It reaches, on its descent, a height of 6.5 m at the instant $t = 22$ sec. The elapsed time is therefore 2 sec.

3.3 Absolute value function

ACTIVITY 1 Absolute value of a real number

On a winter day, the temperature (in °C), recorded at noon, has an absolute value of 5.

a) What is the recorded temperature that day if the temperature is:
1. above 0 °C? ***5 °C*** 2. below 0 °C? ***–5 °C***

b) We represent the absolute value of a number x by $|x|$. Determine:
1. $|+10| =$ ***10*** 2. $|-10| =$ ***10*** 3. $|0| =$ ***0***

c) Is it true to say that two opposite numbers have the same absolute value? ***Yes***

ABSOLUTE VALUE OF A REAL NUMBER

The **absolute value** of a real number a, written $|a|$, is defined by:

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Ex.: $|+4| = 4$; $|-3| = 3$; $|0| = 0$.

Note that the absolute value of a real number is never negative.

1. Determine the following absolute values.

a) $|+8| =$ ***8*** **b)** $|-4.7| =$ ***4.7*** **c)** $|0| =$ ***0*** **d)** $|\pi| =$ π

e) $|-6.53| =$ ***6.53*** **f)** $\left|+\frac{3}{4}\right| =$ $\frac{3}{4}$ **g)** $\left|-\frac{2}{3}\right| =$ $\frac{2}{3}$ **h)** $\left|-\frac{5}{18}\right| =$ $\frac{5}{18}$

ACTIVITY 2 Properties

Consider a real number a and a non-zero real number b. Answer true or false.

a) $|a| \geqslant 0$ ***True*** **b)** $|a| = |-a|$ ***True***

c) $|a + b| = |a| + |b|$ ***False*** **d)** $|a - b| = |a| - |b|$ ***False***

e) $|a \cdot b| = |a| \cdot |b|$ ***True*** **f)** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ ***True***

PROPERTIES

For any real number a and any real number b, we have the following properties.

- $|a| \geq 0$ — Ex.: $|+5| \geq 0$; $|-4| \geq 0$
- $|a| = |-a|$ — $|4| = |-4|$
- $|a + b| \leq |a| + |b|$ — $|7 + (-2)| \leq |7| + |-2|$
- $|a - b| \geq |a| - |b|$ — $|5 - (-3)| \geq |5| - |-3|$
- $|ab| = |a| \cdot |b|$ — $|8 \times (-3)| = |8| \times |-3|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $(b \neq 0)$ — $\left|\frac{-8}{2}\right| = \frac{|-8|}{|2|}$

2. Complete the following using the appropriate symbol $=, >, <$.

a) $|x + 5|$ ___>___ 0 **b)** $|x - 3|$ ___=___ $|3 - x|$ **c)** $|2(x - 1)|$ ___=___ $2|x - 1|$

d) $|7 - 12|$ ___>___ $|7| - |12|$ **e)** $\left|\frac{x+2}{x-1}\right|$ ___=___ $\frac{|x+2|}{|x-1|}$ **f)** $|-6 + 9|$ ___<___ $|-6| + |9|$

ACTIVITY 3 Absolute value equations

a) Today's temperature x, in degrees Celsius, recorded at noon has an absolute value of 20. Determine this temperature if

1. it is warm. ***20°***
2. it is cold. ***-20°***

b) What are the solutions to the equation $|x| = 20$? ***-20 and 20***

c) Consider the equation $|x| = 0$. What is the unique real number that verifies this equation? ***0***

d) Consider the equation $|x| = -4$. Is there a real number that verifies this equation? Justify your answer.

No, since the absolute value of a real number is never negative.

ABSOLUTE VALUE EQUATIONS

The number of solutions to the equation:

$$|x| = k$$

depends on the sign of k.

If $k > 0$	If $k = 0$	If $k < 0$
The equation has 2 solutions.	The equation has 1 solution.	The equation has no solution.
$x = -k$ or $x = k$	$x = 0$	
Ex.: $\lvert x\rvert = 3$	Ex.: $\lvert x\rvert = 0$	Ex.: $\lvert x\rvert = -5$
$S = \{-3, 3\}$	$S = \{0\}$	$S = \varnothing$

3. Solve the following equations.

a) $|x| = 12$
$S = \{-12, 12\}$

b) $|x| = -8$
$S = \varnothing$

c) $|x + 5| = 0$
$S = \{-5\}$

d) $|2x + 1| = 7$
$S = \{-4, 3\}$

e) $\left|\frac{1}{2}x - 5\right| = 4$
$S = \{2, 18\}$

f) $|6 - x| = -3$
$S = \varnothing$

4. Solve the following equations.

a) $2|x - 5| - 4 = 0$
$S = \{3, 7\}$

b) $-2|3x - 1| + 4 = -6$
$S = \left\{-\frac{4}{3}, 2\right\}$

c) $12 - |6 - 2x| = 3$
$S = \left\{-\frac{3}{2}, \frac{15}{2}\right\}$

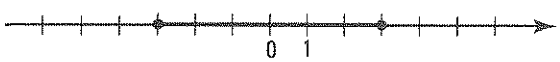
d) $|x - 5| + 8 = 2$
$S = \varnothing$

e) $-3|2x + 5| + 6 = 6$
$S = \left\{-\frac{5}{2}\right\}$

f) $|4x - 5| + 6 = 9$
$S = \left\{\frac{1}{2}, 2\right\}$

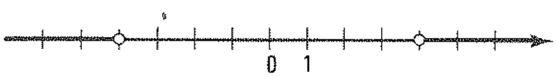
ACTIVITY 4 Absolute value inequalities

a) Consider the inequality $|x| \leqslant 3$.
On the real number line below, represent the set of all real numbers verifying this inequality and find the solution set.



S = $[-3, 3]$

b) Consider the inequality $|x| > 4$.
On the real number line below, represent the set of all real numbers verifying this inequality and find the solution set.



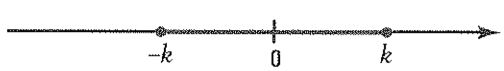
S = $]-\infty, -4[\cup]4, +\infty[$

ABSOLUTE VALUE INEQUALITIES

Given a positive real number k, we have:

$|x| \leqslant k$

$\Leftrightarrow x \geqslant -k$ **and** $x \leqslant k$

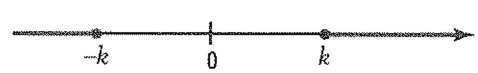


$S = [-k, k]$

Ex.: The inequality $|x| \leqslant 5$ has the solution set: $S = [-5, 5]$.

$|x| \geqslant k$

$\Leftrightarrow x \leqslant -k$ **or** $x \geqslant k$

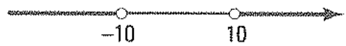


$S =]-\infty, -k] \cup [k, +\infty[$

Ex.: The inequality $|x| \geqslant 5$ has the solution set: $S =]-\infty, -5] \cup [5, +\infty[$.

5. For each of the following inequalities, determine the solution set and represent it on the real number line.

a) $|x| > 10$



$S =]-\infty, -10[\cup]10, +\infty[$

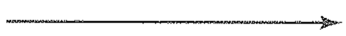
b) $|x| \leq 4$

$S = [-4, 4]$

c) $|x| > -3$

$S = \mathbb{R}$

d) $|x| \leq -2$



$S = \varnothing$

e) $|x| \geq 0$

$S = \mathbb{R}$

f) $|x| \leq 0$

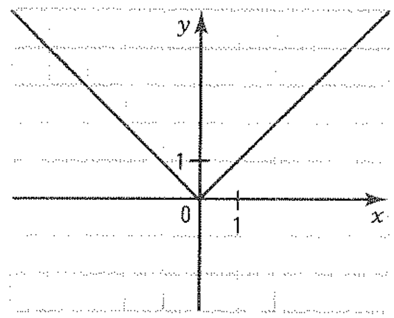
$S = \{0\}$

ACTIVITY 5 Basic absolute value function

Consider the function f defined by the rule $y = |x|$.

a) Complete the following table of values.

	–3	–2	–1	0	1	2	3
	3	*2*	*1*	*0*	*1*	*2*	*3*



b) Represent the function f in the Cartesian plane.

c) Determine

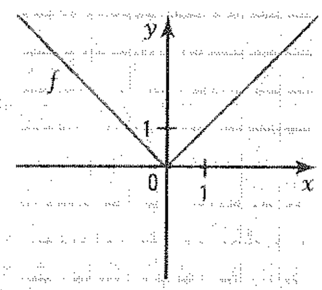
1. dom f. $\mathbb{R}$
2. ran f. $\mathbb{R}_+$
3. the zero of f. *0*
4. the initial value of f. *0*
5. the sign of f. $f(x) \geq 0$ over $\mathbb{R}$. $f(x) \leq 0$, 0
6. the variation of f. $f \nearrow$ over $[0, +\infty[$, $f \searrow$ over $]-\infty, 0]$
7. the extrema of f. $\min f = 0$

BASIC ABSOLUTE VALUE FUNCTION

- The function f defined by the rule:

$$f(x) = |x|$$

is called the basic **absolute value** function.



- We have:

dom $f = \mathbb{R}$ ran $f = \mathbb{R}_+$

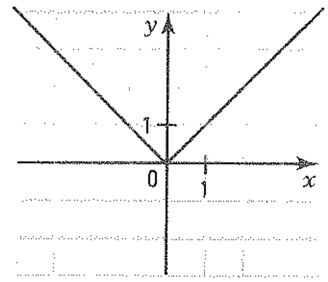
The zero of f is 0. The **initial value** of f is 0.

Sign of f: $f(x) \geq 0$ over $\mathbb{R}$.

Variation of f: f is increasing over $\mathbb{R}_+$; f is decreasing over $\mathbb{R}_-$.

The function f has a **minimum** of 0.

6. Consider the basic absolute value function $f(x) = |x|$ represented on the right.



a) Using the graph, find the values of x for which the function $f(x)$ is:

1. equal to 2. ***−2 and 2***
2. less than 2. ***$x \in]-2, 2[$***
3. less than or equal to 2. ***$x \in [-2, 2]$***
4. greater than 2. ***$x \in]-\infty, -2[\cup]2, +\infty[$***
5. greater than or equal to 2. ***$x \in]-\infty, -2] \cup [2, +\infty[$***

b) Using the graph, solve the following equations or inequalities:

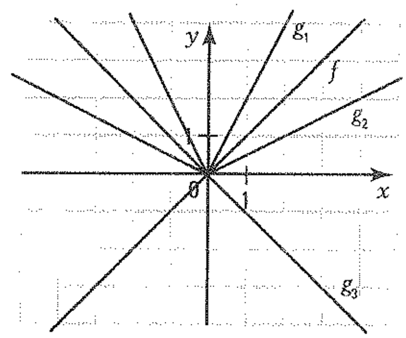
1. $|x| = 1$ ***$S = \{-1, 1\}$*** 2. $|x| = 0$ ***$S = \{0\}$*** 3. $|x| = -1$ ***$S = \varnothing$***

4. $|x| \leqslant 1$ ***$S = [-1, 1]$*** 5. $|x| > 3$ ***$S =]-\infty, -3[\cup]3, +\infty[$*** 6. $|x| > -1$ ***$S = \mathbb{R}$***

ACTIVITY 6 Absolute value function $f(x) = a|b(x - h)| + k$

The basic absolute value function $f(x) = |x|$ can be transformed into an absolute value function defined by the rule

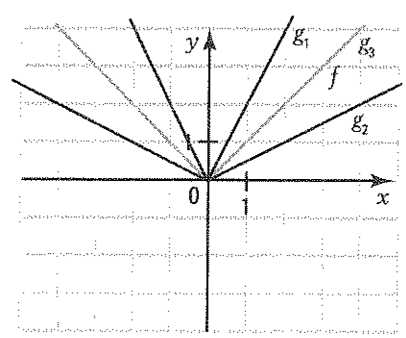
$$\boxed{g(x) = a|b(x - h)| + k}$$



a) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = a|x|$.
Represent, in the same Cartesian plane, the functions $g_1(x) = 2|x|$, $g_2(x) = \frac{1}{2}|x|$ and $g_3(x) = -|x|$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection about the x axis.***
4. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = a|x|$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.
5. Is the graph of $g(x) = a|x|$ open upward or downward when
   1) $a > 0$? ***upward*** 2) $a < 0$? ***downward***

b) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |bx|$.

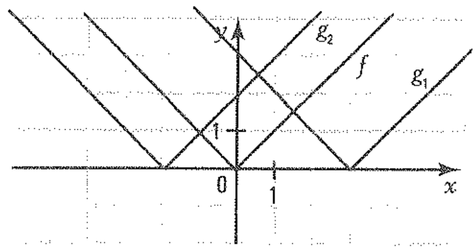


Represent, in the same Cartesian plane, the functions $g_1(x) = |2x|$, $g_2(x) = \left|\frac{1}{2}x\right|$ and $g_3(x) = |-x|$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection about the y axis.***

4. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |bx|$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$
5. Compare the graphs of the functions $y = 2|x|$ and $y = |2x|$ obtained in a) and b). Justify your answer. ***They are the same. In fact, $|2x| = |2| \cdot |x| = 2|x|$.***
6. Compare the graphs $f(x) = |x|$ and $f(x) = |-x|$. Justify your answer.
They are the same. In fact, $|x| = |-x|$.

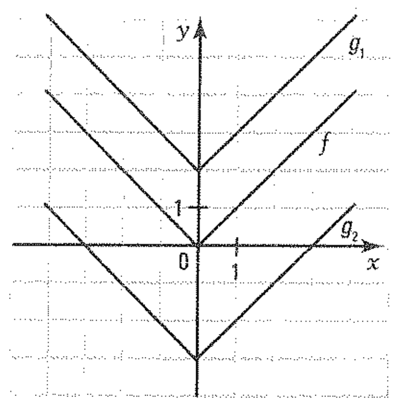
c) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |x - h|$.



Represent, in the same Cartesian plane, the functions $g_1(x) = |x - 3|$ and $g_2(x) = |x + 2|$ and explain how to deduce the graph of g from the graph of f when

1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |x - h|$ by the transformation $(x, y) \rightarrow$ ***$(x + h, y)$***.

d) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |x| + k$.



Represent, in the same Cartesian plane, the functions $g_1(x) = |x| + 2$ and $g_2(x) = |x| - 3$ and explain how to deduce the graph of g from the graph of f when

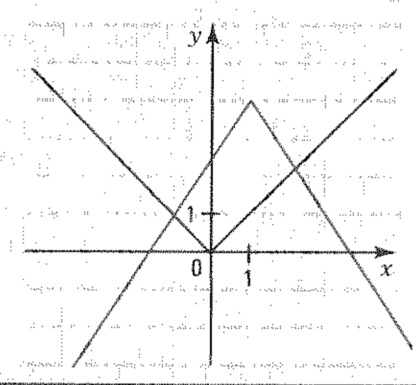
1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |x| + k$ by the transformation $(x, y) \rightarrow$ ***$(x, y + k)$.***

ABSOLUTE VALUE FUNCTION $f(x) = a|b(x - h)| + k$

The graph of the function $f(x) = a|b(x - h)| + k$ is deduced from the graph of the basic absolute value function $y = |x|$ by the transformation:

$$(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$$

Ex.: The graph of the function $f(x) = -3|\frac{1}{2}(x - 1)| + 4$ is deduced from the graph of the basic absolute value function $g(x) = |x|$ by the transformation: $(x, y) \rightarrow (2x + 1, -3y + 4)$.



7. The following functions have a rule of the form $f(x) = a|b(x-h)| + k$.
$f_1(x) = 3|x|$, $f_2(x) = |2x|$, $f_3(x) = |x+4|$, $f_4(x) = |x| + 1$ and $f_5(x) = 2|3(x-1)| - 4$.
Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the function from the basic absolute value function $g(x) = |x|$.

	a	b	h	k	Rule
$f_1(x) = 3\|x\|$	3	1	0	0	$(x, y) \rightarrow (x, 3y)$
$f_2(x) = \|2x\|$	1	2	0	0	$(x, y) \rightarrow \left(\frac{x}{2}, y\right)$
$f_3(x) = \|x+4\|$	1	1	-4	0	$(x, y) \rightarrow (x - 4, y)$
$f_4(x) = \|x\| + 1$	1	1	0	1	$(x, y) \rightarrow (x, y + 1)$
$f_5(x) = 2\|3(x-1)\| - 4$	2	3	1	-4	$(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 4\right)$

8. In each of the following cases, we apply a transformation to the basic absolute value function $y = |x|$. Find the rule of the function obtained by applying the given transformation.

a) $(x, y) \rightarrow (x, -y)$ $y = -|x|$

b) $(x, y) \rightarrow (x - 2, y + 4)$ $y = |x + 2| + 4$

c) $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$ $y = |2x|$

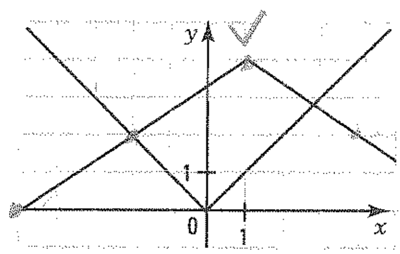
d) $(x, y) \rightarrow (5x, y)$ $y = \left|\frac{x}{5}\right|$

e) $(x, y) \rightarrow (3x, -7y)$ $y = -7\left|\frac{1}{3}x\right|$

f) $(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 4\right)$ $y = 2|3(x - 1)| - 4$

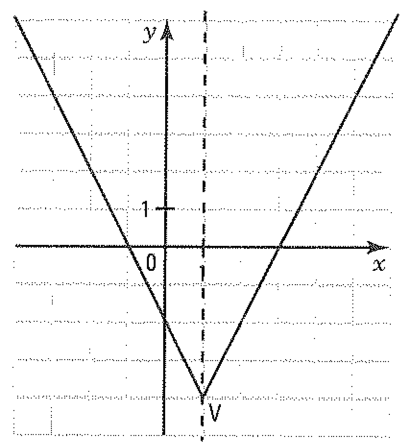
9. From the basic absolute value function and using the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$, represent the function $y = -2\left|\frac{1}{3}(x-1)\right| + 4$ in the Cartesian plane.

For example, (1, 1) → (4, 2)



ACTIVITY 7 Graphing the function $f(x) = a|b(x-h)| + k$

Consider the function $f(x) = 4\left|-\frac{1}{2}(x-1)\right| - 4$.



a) Identify the parameters a, b, h and k.

$a = 4$, $b = -\frac{1}{2}$, $h = 1$ and $k = -4$

b) Is the graph open upward or downward? Justify your answer.

Upward, $a > 0$.

c) What are the coordinates of the vertex? *V(1, −4)*

d) Find the zeros of the function. *−1 and 3*

e) Represent the function f in the Cartesian plane after completing the following table of values.

x	−2	1	4
y	2	−4	2

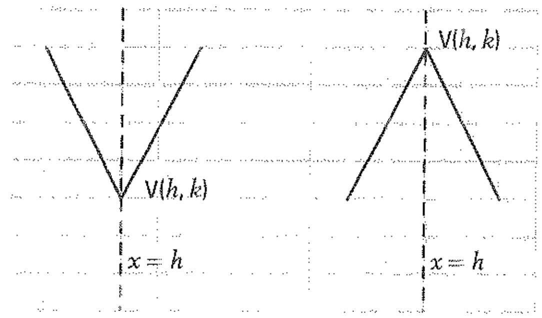
GRAPH OF AN ABSOLUTE VALUE FUNCTION

Consider the **absolute value function** defined by the rule:

$$f(x) = a|b(x - h)| + k$$

- The graph is open:
  - **upward** if a > 0.
  - **downward** if a < 0.
- The graph has the vertex: $V(h, k)$
- The graph has the following line as an axis of symmetry:

$$x = h$$



10. Write the rules of the following functions in the form $y = a|x - h| + k$ and identify the parameters a, h and k.

a) $y = -2|3x + 3| + 5$

$y = -6|x + 1| + 5$; $a = -6$, $h = -1$, $k = 5$

b) $y = 4|6 - 3x| + 5$

$y = 12|x - 2| + 5$; $a = 12$, $h = 2$, $k = 5$

c) $y = -\frac{1}{2}|8x - 4| + 3$

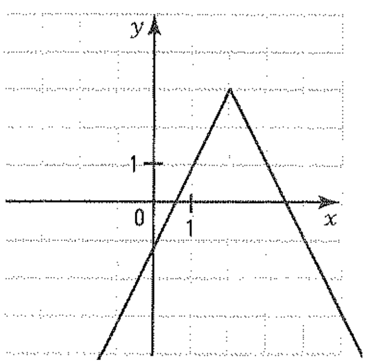
$y = -4\left|x - \frac{1}{2}\right| + 3$; $a = -4$, $h = \frac{1}{2}$, $k = 3$

d) $y = -\frac{5}{6}\left|4 - \frac{1}{5}x\right| + 3$

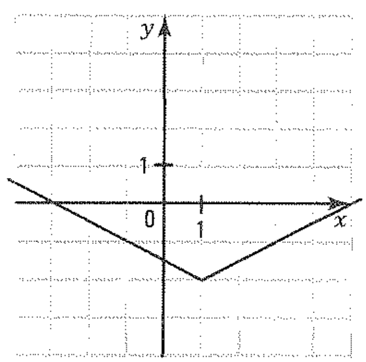
$y = -\frac{1}{6}|x - 20| + 3$; $a = -\frac{1}{6}$, $h = 20$, $k = 3$

11. Graph the following functions.

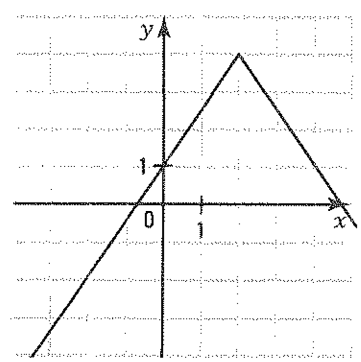
a) $y = -2|x - 2| + 3$



b) $y = \frac{1}{8}|4 - 4x| - 2$



c) $y = -\frac{1}{2}|3x - 6| + 4$

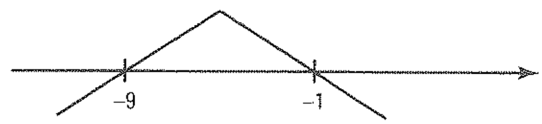


ACTIVITY 8 Determining the sign of an absolute value function

Consider the absolute value function $f(x) = -2|x + 5| + 8$.

a) What are the zeros of this function? ***−9 and −1***

b) Determine the sign of this function using a sketch.

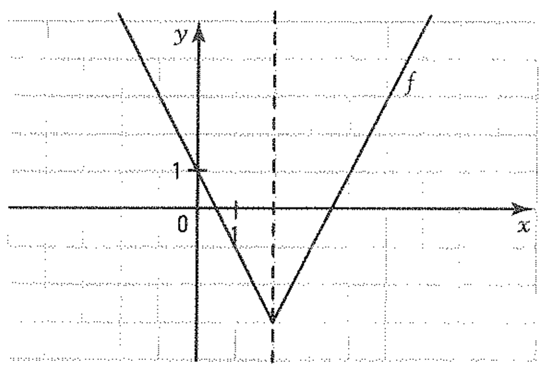


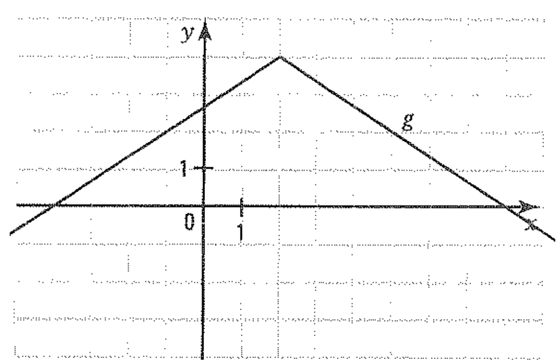
$f(x) \geq 0$ if $x \in [-9, -1]$ and $f(x) \leq 0$ if $x \in\ ]-\infty, -9] \cup [-1, +\infty[$

ACTIVITY 9 Study of an absolute value function

Consider the functions $f(x) = \frac{1}{2}|8 - 4x| - 3$ and $g(x) = -\frac{1}{3}|2x - 4| + 4$.

a) Write each of the rules in the form $y = a|x - h| + k$ and represent the functions in the Cartesian plane.





$f(x) = 2|x - 2| - 3$ ***$g(x) = -\frac{2}{3}|x - 2| + 4$***

b) Do a study of each of the preceding functions and complete the table below.

Properties	f	g
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	***$[-3, +\infty[$***	***$]-\infty, 4]$***
Zeros	***$\frac{1}{2}$ and $\frac{7}{2}$***	***−4 and 8***
Initial value	***1***	***$\frac{8}{3}$***
Sign	***$f(x) \geq 0$ over $]-\infty, \frac{1}{2}] \cup [\frac{7}{2}, +\infty[$*** ***$f(x) \leq 0$ over $[\frac{1}{2}, \frac{7}{2}]$***	***$f(x) \geq 0$ over $[-4, 8]$*** ***$f(x) \leq 0$ over $]-\infty, -4] \cup [8, +\infty[$***
Variation	***$f \searrow$ over $]-\infty, 2]$; $f \nearrow$ over $[2, +\infty[$***	***$f \nearrow$ over $]-\infty, 2]$; $f \searrow$ over $[2, +\infty[$***
Extrema	***min $f = -3$***	***max $f = 4$***

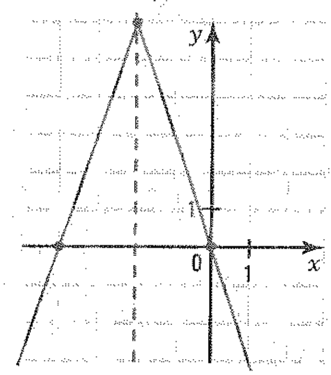
STUDY OF AN ABSOLUTE VALUE FUNCTION

Given the absolute value function: $f(x) = a|b(x - h)| + k$, we have:

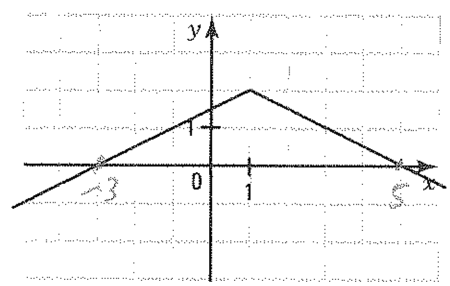
- dom $f = \mathbb{R}$.
- ran $f = [k, +\infty[$ if $a > 0$; $[-\infty, k[$ if $a < 0$.
- The zero(s) of f exist if a and k are opposite signs or if $k = 0$.
- To study the sign of f,
  - we find the zero(s) if they exist;
  - we establish the sign of f from a sketch of the graph.
- **Variation**
  If $a > 0$, f is decreasing over $]-\infty, h]$. If $a < 0$, f is increasing over $]-\infty, h]$.
  f is increasing over $[h, +\infty[$. f is decreasing over $[h, +\infty[$.
- **Extrema**
  If $a > 0$, f has a minimum. min $f = k$.
  If $a < 0$, f has a minimum. max $f = k$.

Ex.: Consider the function $f(x) = -3|x + 2| + 6$. ($a = -3, b = 1, h = -2, k = 6$)

- Open downward, $a < 0$.
- Vertex: V(–2, 6).
- Axis of symmetry: $x = -2$.
- Zeros: $-3|x + 2| + 6 = 0$
  $|x + 2| = 2$
  $\Leftrightarrow \quad x + 2 = -2 \quad$ or $x + 2 = 2$
  $x = -4 \quad$ or $\quad x = 0$
- Initial value: $y = 0$.
- dom $f = \mathbb{R}$; ran $f =]-\infty, 6]$
- Sign of f: $f(x) \geqslant 0$ over $[-4, 0]$; $f(x) \leqslant 0$ over $]-\infty, -4] \cup [0, +\infty[$.
- Variation of f: f is increasing over $]-\infty, -2]$; f is decreasing over $[-2, +\infty[$.
- max $f = 6$.



12. Represent the graph and do a study of the function
$f(x) = -\frac{1}{4}|2(x - 1)| + 2$.



dom = $\mathbb{R}$; ran = $]-\infty, 2]$.
Zeros: –3 and 5.
Initial value: 1.5.
Sign: $f(x) \geqslant 0$ over $[-3, 5]$.
$f(x) < 0$ over $]-\infty, 2[\cup]5, +\infty[$.
Variation: f ↗ over $]-\infty, 1]$; f ↘ over $[1, +\infty[$
Extrema: max = 2

13. Determine the domain and range of each of the following functions.

a) $y = -2|x + 5| - 1$

dom $= \mathbb{R}$, *ran* $=]-\infty, -1]$

b) $y = \frac{1}{4}|-2(x - 1)| + 5$

dom $= \mathbb{R}$, *ran* $= [5, +\infty[$

14. Determine the zeros of the following functions.

a) $y = 3|x - 5| - 6$ — **3 and 7**

b) $y = -\frac{1}{2}|6 - 3x| + 4$ — $-\frac{2}{3}$ **and** $\frac{14}{3}$

c) $y = 4|2x + 1| + 8$ — **No zero**

d) $y = -5|6 - x|$ — **6**

15. Consider the linear function $f(x) = 2x - 3$ and the absolute value function $g(x) = 3|3x + 5| - 4$. Determine the initial value of the composite

a) $g \circ f$ — **8**

b) $f \circ g$ — **19**

16. Determine the interval over which each of the following functions is positive.

a) $y = -\frac{1}{3}|x - 5| + 2$

$f(x) \geqslant 0$ over $[-1, 11]$

b) $y = 2|3 - 2x| - 4$

$f(x) \geqslant 0$ over $\left]-\infty, \frac{1}{2}\right] \cup \left[\frac{5}{2}, +\infty\right[$

c) $y = \frac{3}{4}|-2x + 4| - 3$

$f(x) \geqslant 0$ over $]-\infty, 0] \cup [4, +\infty[$

d) $y = 3|x - 5| + 6$

$f(x) \geqslant 0$ over $\mathbb{R}$

17. Determine the interval over which each of the following functions is increasing.

a) $y = 5|6 - 4x| + 2$

$f \nearrow$ over $\left[\frac{3}{2}, +\infty\right[$

b) $y = -3|2x + 4| + 5$

$f \nearrow$ over $]-\infty, -2]$

18. Determine the solution set to each of the following inequalities.

a) $|x - 5| > 3$

$S =]-\infty, 2[\cup]8, +\infty[$

b) $|6 - x| \leqslant 1$

$S = [5, 7]$

c) $|3x - 2| \geqslant 4$

$S = \left]-\infty, -\frac{2}{3}\right] \cup [2, +\infty[$

d) $|2x + 5| \leqslant 0$

$S = \left\{-\frac{5}{2}\right\}$

e) $-2|x + 1| + 5 > -5$

$S =]-6, 4[$

f) $3|2 - x| + 4 > 1$

$S = \mathbb{R}$

g) $6 - 3|x - 1| \leqslant 0$

$S =]-\infty, -1] \cup [3, +\infty[$

h) $-|2x - 1| + 5 > 0$

$S =]-2, 3[$

i) $\left|\frac{x}{2} - 1\right| > 0$

$S = \mathbb{R} \setminus \{2\}$

19. Study each of the following functions and complete the following table.

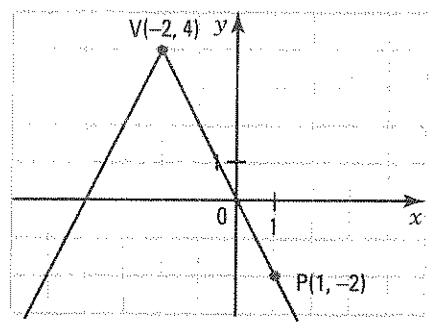
	$f(x) = -2\lvert x - 1\rvert + 4$	$f(x) = 3\lvert x + 2\rvert - 6$	$f(x) = \frac{1}{2}\lvert x - 4\rvert + 3$	$f(x) = -3\lvert 5 - x\rvert$
Dom f	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Ran f	*]−∞, 4]*	*[−6, +∞[*	*[3, +∞[*	*]−∞, 0]*
Zero(s) if they exist	*−1 and 3*	*−4 and 0*	*None*	*5*
Initial value	*2*	*0*	*5*	*−15*
Sign	*f(x) ≥ 0 over [−1, 3]* *f(x) < 0 over]−∞, −1[∪]3, +∞[*	*f(x) ≥ 0 over]−∞, −4] ∪ [0, +∞[* *f(x) < 0 over]−4, 0[*	*f(x) ≥ 0 over ℝ* *f(x) < 0 never*	*f(x) ≥ 0 over {5}* *f(x) < 0 over ℝ \ {5}*
Variation	*f ↗ over]−∞, 1]* *f ↘ over [1, +∞[*	*f ↗ over [−2, +∞[* *f ↘ over]−∞, −2]*	*f ↗ over [4, +∞[* *f ↘ over]−∞, 4]*	*f ↗ over]−∞, 5]* *f ↘ over [5, +∞[*
Extrema	*max = 4*	*min = −6*	*min = 3*	*max = 0*

ACTIVITY 10 Finding the rule of an absolute value function

The rule of any absolute value function can be written in the form $f(x) = a|x - h| + k$.

a) Consider the function $f(x) = 3|-2(x - 5)| + 7$.
Write the rule of this function in the form $f(x) = a|x - h| + k$. *$y = 6|x - 5| + 7$*

b) Consider the absolute value function with the vertex V(−2, 4) and passing through the point P(1, −2).



1. Identify h and k. *$h = -2$, $k = 4$*
2. Determine a knowing that the coordinates of the point P(1, −2) verify the rule of the function.
   We have: $y = a|x + 2| + 4$
   $-2 = a|1 + 2| + 4$
   $-6 = 3a$
   $a = -2$
3. What is the rule of the function? *$f(x) = -2|x + 2| + 4$*

FINDING THE RULE OF AN ABSOLUTE VALUE FUNCTION

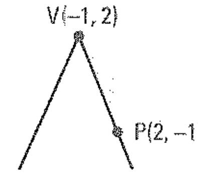
The rule of any absolute value function can be written in the form:

$$f(x) = a|x - h| + k$$

1st case: The vertex V and a point P are given.

1. Identify the parameters h and k.
2. Find a after replacing x and y in the rule by the coordinates of the given point P.
3. Deduce the rule.

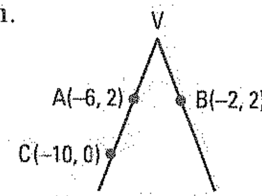
1. $h = -1$ and $k = 2$
   $y = a|x + 1| + 2$
2. $-1 = a|2 + 1| + 2$
   $-1 = 3a + 2$
   $a = -1$
3. $y = -|x + 1| + 2$



2nd case: Three points, of which two have the same y-coordinate, are given.

1. Identify h as half the sum of the x-coordinates of the points with the same y-coordinates.
2. Find the slope of the ray passing through two given points, and establish parameter a according to the opening of the graph.
3. Find k after replacing x and y in the rule by the coordinates of one of the given points.
4. Deduce the rule.

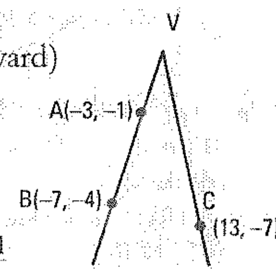
1. $h = \frac{(-6) + (-2)}{2} = -4$
2. Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{2}$
   $a = -\frac{1}{2}$ (open downward)
3. $y = -\frac{1}{2}|x + 4| + k$
   $2 = -\frac{1}{2}|-2 + 4| + k$
   $k = 3$
4. $y = -\frac{1}{2}|x + 4| + 3$



3rd case: Any three points are given.

1. Find the slope of the ray passing through two given points, and establish parameter a according to the opening of the graph.
2. Find the equation of each ray knowing that their slopes are opposite.
3. Find the coordinates (h, k) of the vertex V, which is the intersection of the two rays.
4. Deduce the rule.

1. Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{4}$
   $a = \frac{-3}{4}$ (open downward)
2. $y_1 = \frac{3}{4}x + \frac{5}{4}$
   $y_2 = -\frac{3}{4}x + \frac{11}{4}$
3. $\frac{3}{4}x + \frac{5}{4} = -\frac{3}{4}x + \frac{11}{4}$
   $6x = 6$
   $x = 1 \Rightarrow y = 2$
   Thus, V(1, 2)
4. $y = -\frac{3}{4}|x - 1| + 2$



20. Find the rule of an absolute value function whose graph

a) has the vertex V(3, 4) and passes through the point P(7, 6). *$y = \frac{1}{2}|x - 3| + 4$*

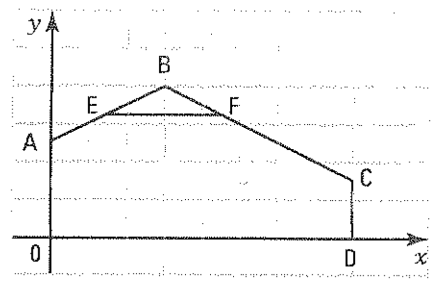
b) passes through the points A(2, –6), B(5, –8) and C(–4, –6). *$y = -\frac{2}{3}|x + 1| - 4$*

c) passes through the points A(1, –1), B(3, –5) and C(–4, –3). *$y = -2|x + 1| + 3$*

21. In order to draw the simulated trajectory of a toy airplane, Ethan uses the rule of an absolute value function that gives the airplane's height y, in metres, as a function of elapsed time x, in seconds. The rule of the function is $y = -\frac{5}{4}|x - 8| + 10$.

For how many seconds is the height of the airplane above 7 m? *4.8 seconds*

22. In the Cartesian plane on the right, a view of an airplane hangar is represented with the roof of the hangar corresponding to an absolute value function given by the rule $y = -\frac{1}{2}|x - 6| + 8$.

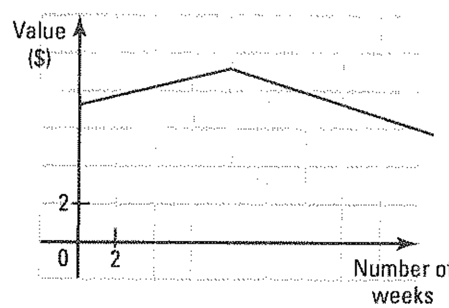


a) What is the height of the wall A0? *5 m*

b) What is the height of the wall CD if the width of the hangar is equal to 16 m? *3 m*

c) The ceiling EF is built at a height of 6.5 m. What is the width of the ceiling? ~~*5.6 m*~~ 6 m ?

23. The graph on the right represents the evolution of a share's value on the stock market. Eight weeks after its purchase, the share reaches its maximum value of \$9. If it initially was worth \$7, what will it be worth after 13 weeks?



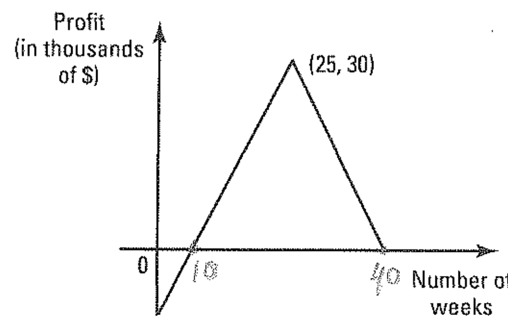
$y = a|x - 8| + 9$; $7 = 8a + 9$; $a = -\frac{1}{4}$

$y = -\frac{1}{4}|x - 8| + 9$.

It will be worth \$7.75.

24. The graph on the right represents the profit of a recycling company during its first 40 weeks of operation.

During how many weeks was the profit greater than \$15 000?



$y = -2|x - 25| + 30$

$-2|x - 25| + 30 = 15$; $x = 17.5$ or $x = 32.5$.

During 15 weeks.

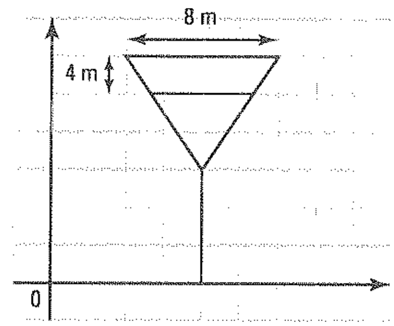
25. The air conditioning system in an office building has been programmed so that it turns on when the outside temperature reaches 23 °C and turns off when it reaches 20 °C.

The outside temperature varies according to the rule of the absolute value function given by $y = -3|x - 6| + 35$ where x represents the elapsed number of hours since 6 a.m. and y represents the outside temperature in °C.

How many hours was the system on?

It turns on at 8 a.m. and turns off at 5 p.m. The system is on during 9 hours.

26. The lateral view of a channelling system is represented in the Cartesian plane on the right, scaled in metres.



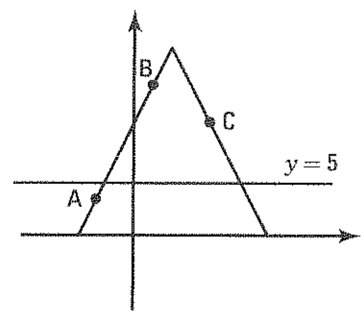
The walls of this system are represented by an absolute value function with the rule: $y = 3|x - 8| + 12$.

A filtering net is placed 4 m below the ceiling of the canal. If the width of the canal is 8 m, what is the width of the filtering net?

When $x = 12$, $y = 24$;

When $y = 20$, $x = \frac{16}{3}$ or $x = \frac{32}{3}$. The width of the net is 5.33 m.

27. The graph on the right represents the front of a house. The base of the roof corresponds to the line $y = 5$.



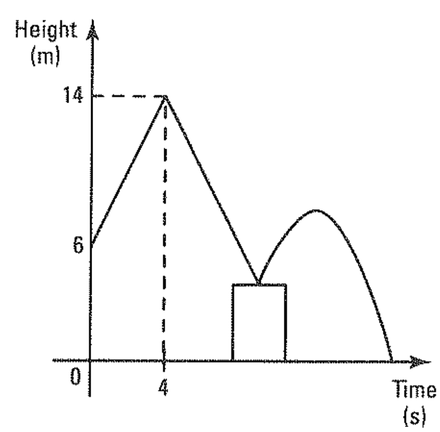
The sides of the roof form the graph of an absolute value function passing through the points A(–2, 3), B(2, 13) and C(8, 8).

What is the area of the triangle limited by the roof and the line?

$y = -\frac{5}{2}|x - 4| + 18$; base $= \frac{52}{5}$; height $= 13$.

The area of the triangle is 67.6 u^2.

28. A projectile is thrown from a height of 6 m and follows the trajectory of an absolute value function. It reaches a maximum height of 14 m after 4 seconds. Five seconds after reaching its maximum height, it bounces off a cement block and follows the trajectory of a quadratic function. If the maximum height of the second bounce is 8 m and occurs three seconds after bouncing off the cement block, when will the projectile hit the ground? (Round your answer to the nearest second).



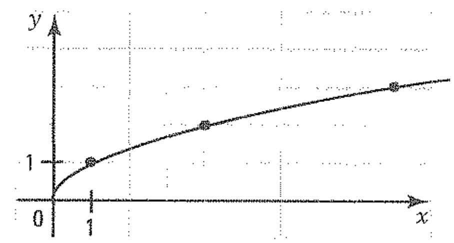
$y = -2|x - 4| + 14$, $P(9, 4)$; $y = -\frac{4}{9}(x - 12)^2 + 8$;

The projectile hits the ground at $t = 16$ s.

3.4 Square root function

Activity 1 Square root function

Consider the function f defined by the rule $f(x) = \sqrt{x}$.



a) What condition must be placed on x for $\sqrt{x}$ to exist in $\mathbb{R}$?

x must be positive or zero

b) Complete the following table of values.

x	0	1	4	9
y	***0***	***1***	***2***	***3***

c) Represent the function f in the Cartesian plane.

d) Determine

1. dom $f =$ $\mathbb{R}_+$
2. ran $f =$ $\mathbb{R}_+$
3. the zero of f. ***0***
4. the initial value of f. ***0***
5. the sign of f. ***$f(x) \geqslant 0$ over $\mathbb{R}_+$.***
6. the variation of f. ***$f \nearrow$ over $\mathbb{R}_+$.***
7. the extrema of f. ***min $f = 0$***

BASIC SQUARE ROOT FUNCTION

- The function f defined by the rule:

$$f(x) = \sqrt{x}$$

is called the **basic square root function.**

- We have:

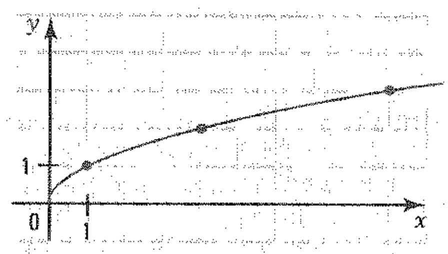
dom $f = \mathbb{R}_+$. ran $f = \mathbb{R}_+$.

The zero of f is 0. The initial value is 0.

Sign of f: $f(x) \geqslant 0, \forall x \in$ dom f.

Variation of f: f is increasing, $\forall x \in$ dom f.

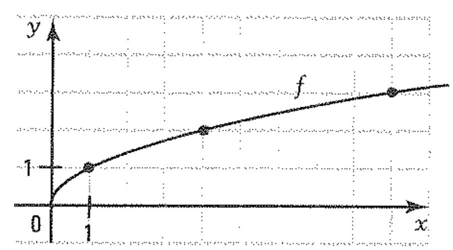
The function f has a minimum equal to 0.



The point O (0, 0) is the vertex of the function.

1. Consider the basic square root function $f(x) = \sqrt{x}$ represented on the right.

Using the graph, find the values of x for which



a) $f(x) = 3$ ***$x = 9$***

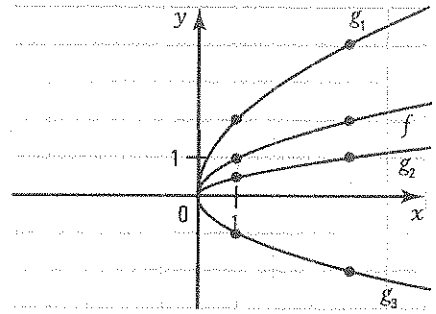
b) $f(x) \geqslant 1$ ***$x \in [1, +\infty[$***

c) $0 \leqslant f(x) < 2$ ***$x \in [0, 4[$***

d) $f(x) < 0$ ***None since the function is never strictly negative.***

e) $1 < f(x) < 3$ ***$x \in]1, 9[$***

ACTIVITY 2 Square root function $f(x) = a\sqrt{b(x-h)} + k$



The basic square root function $f(x) = \sqrt{x}$ can be transformed into a square root function defined by the rule

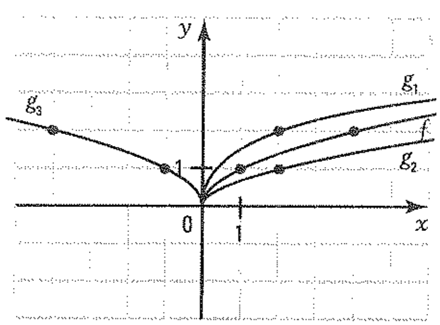
$$\boxed{g(x) = a\sqrt{b(x-h)} + k}.$$

a) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = a\sqrt{x}$.

Represent, in the same Cartesian plane, the functions $g_1(x) = 2\sqrt{x}$, $g_2(x) = \frac{1}{2}\sqrt{x}$ and $g_3(x) = -\sqrt{x}$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection about the x axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = a\sqrt{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.
5. Is the graph of $g(x) = a\sqrt{x}$ located in the 1st or 4th quadrant when
   1) $a > 0$? ***1st quadrant*** 2) $a < 0$? ***4th quadrant***

b) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{bx}$.

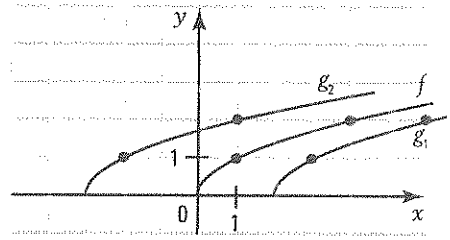


Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{2x}$, $g_2(x) = \sqrt{\frac{1}{2}x}$ and $g_3(x) = \sqrt{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection about the y axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{bx}$ by the transformation $(x, y) \rightarrow$ ***$\left(\frac{x}{b}, y\right)$***
5. In which quadrant is the graph of $g(x) = \sqrt{bx}$ when
   1) $b > 0$? ***1st quadrant*** 2) $b < 0$? ***2nd quadrant***
6. What can you say about the graph of the function $y = 2\sqrt{x}$ and that of the function $y = \sqrt{4x}$? Justify your answer.
   They are the same. In fact, $\sqrt{4x} = \sqrt{4} \cdot \sqrt{x} = 2\sqrt{x}$ (property of radicals).

c) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x - h}$.

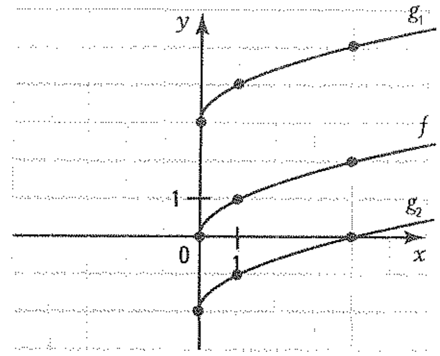
Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x-2}$ and $g_2(x) = \sqrt{x+3}$ and explain how to deduce the graph of g from the graph of f when



1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x-h}$ by the transformation $(x, y) \rightarrow$ ***(x + h, y)***

d) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x} + k$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x} + 3$ and $g_2(x) = \sqrt{x} - 2$ and explain how to deduce the graph of g from the graph of f when



1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x} + k$ by the transformation $(x, y) \rightarrow$ ***(x, y + k)***

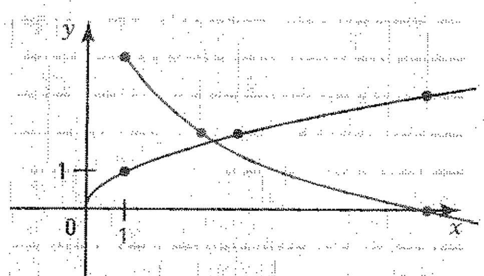
SQUARE ROOT FUNCTION $f(x) = a\sqrt{b(x - h)} + k$

The graph of the function $f(x) = a\sqrt{b(x - h)} + k$ is deduced from the graph of the basic square root function $y = \sqrt{x}$ by the transformation

$$(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$$

Ex.: The basic square root function $y = \sqrt{x}$ and the square root function

$y = -2\sqrt{\frac{1}{2}(x - 1)} + 4$ are represented on the right.



The rule of the transformation applied to the graph of the basic square root function is:
$(x, y) \rightarrow (2x + 1, -2y + 4)$.

Mapping Rule

2. The following functions have a rule of the form $f(x) = a\sqrt{b(x-h)} + k$.

$f_1(x) = 3\sqrt{x}$, $f_2(x) = \sqrt{2x}$, $f_3(x) = \sqrt{x+4}$, $f_4(x) = \sqrt{x} + 1$ and $f_5(x) = 2\sqrt{3(x-1)} - 4$.

Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the function from the basic function $g(x) = \sqrt{x}$.

	a	b	h	k	Rule
$f_1(x) = 3\sqrt{x}$	3	1	0	0	$(x, y) \to (x, 3y)$
$f_2(x) = \sqrt{2x}$	1	2	0	0	$(x, y) \to \left(\frac{x}{2}, y\right)$
$f_3(x) = \sqrt{x+4}$	1	1	-4	0	$(x, y) \to (x - 4, y)$
$f_4(x) = \sqrt{x} + 1$	1	1	0	1	$(x, y) \to (x, y + 1)$
$f_5(x) = 2\sqrt{3(x-1)} - 4$	2	3	1	-4	$(x, y) \to \left(\frac{x}{3} + 1, 2y - 4\right)$

3. In each of the following cases, we apply a transformation to the basic square root function $f(x) = \sqrt{x}$. Find the rule of the function obtained by applying the given transformation.

a) $(x, y) \to (-x, y)$ $y = \sqrt{-x}$

b) $(x, y) \to (x - 5, y + 2)$ $y = \sqrt{x + 5} + 2$

c) $(x, y) \to \left(\frac{x}{5}, y\right)$ $y = \sqrt{5x}$

d) $(x, y) \to (x, -4y)$ $y = -4\sqrt{x}$

e) $(x, y) \to (2x, -6y)$ $y = -6\sqrt{\frac{1}{2}x}$

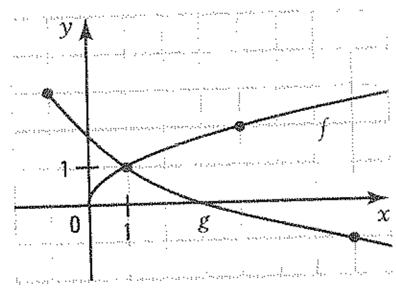
f) $(x, y) \to \left(\frac{x}{4} + 3, 3y - 5\right)$ $y = 3\sqrt{4(x-3)} - 5$

4. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = -2\sqrt{\frac{1}{2}(x+1)} + 3$.

a) Give the rule of the transformation which enables you to obtain the graph of g from the graph of f.

$(x, y) \to (2x - 1, -2y + 3)$

b) Draw the graph of g from the graph of f.



ACTIVITY 3 Graph of a square root function

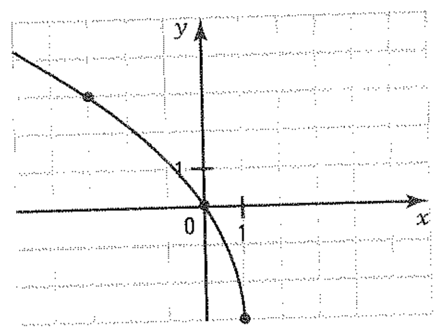
Consider the function $f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3$.

a) Identify the parameters a, b, h and k.

$a = 6$, $b = -\frac{1}{4}$, $h = 1$ and $k = -3$

b) Write the rule of the function in the form $f(x) = a\sqrt{-(x-h)} + k$.

$f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3 = 6\sqrt{\frac{1}{4}}\sqrt{-(x-1)} - 3 = 3\sqrt{-(x-1)} - 3$



c) What are the coordinates of the vertex? $V(1, -3)$

d) Represent the function f in the Cartesian plane after completing the following table of values.

x	−3	0	1
y	3	0	-3

e) What is the zero of f? 0

ACTIVITY 4 Finding the zero of a square root function

a) Consider the function with the rule: $y = -2\sqrt{3(x+1)} + 6$.

Justify the steps in finding the zero of this function.

$-2\sqrt{3(x+1)} + 6 = 0$ *Replace y by 0.*

$\sqrt{3(x+1)} = 3$ *Isolate the square root.*

$3(x+1) = 9$ *Square each side of the equality.*

$x + 1 = 3$ *Divide each side by 3.*

$x = 2$ *Subtract 1 from each side.*

b) Under what conditions does the zero of a function $y = a\sqrt{b(x-h)} + k$ exist?

If a and k are opposite signs or if k = 0.

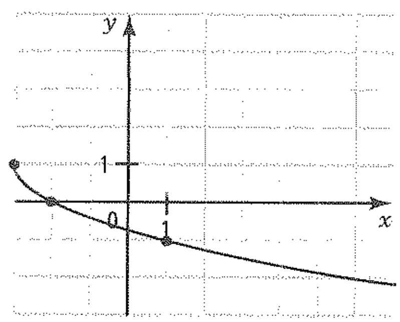
ACTIVITY 5 Study of a square root function

Consider the function f with rule $y = -\frac{1}{2}\sqrt{4x + 12} + 1$.

a) Write the rule in the form $y = a\sqrt{x-h} + k$.

$y = \frac{-1}{2}\sqrt{4(x+3)} + 1 = -\sqrt{x+3} + 1$

b) Graph the function f.



c) Determine

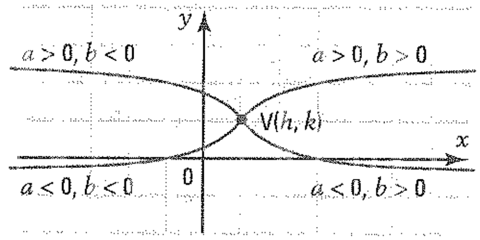
1. dom $f =$ $[-3, +\infty[$
2. ran $f =$ $]-\infty, 1]$
3. the zero of f (if it exists). -2
4. the initial value of f. $-\sqrt{3} + 1$
5. the sign of f. $f(x) \geq 0$ over $[-3, -2]$; $f(x) \leq 0$ over $[-2, +\infty[$
6. the variation of f. $f \nearrow$ never; $f \searrow$ over $[-3, +\infty[$
7. the extrema of f. max $f = 1$

STUDY OF A SQUARE ROOT FUNCTION

Consider the square root f with the rule:

$$f(x) = a\sqrt{b(x-h)} + k$$

We have the following four cases:



- dom $f = [h, +\infty[$ if $b > 0$; ran $f = [k, +\infty[$ if $a > 0$;
  dom $f =]-\infty, h]$ if $b < 0$. ran $f =]-\infty, k]$ if $a < 0$.
- The zero of f exists if a and k are opposite signs or if $k = 0$.
- To study the sign of f,
  - we find the zero (if it exists);
  - we establish the sign of f from a sketch of the graph.
- **Variation**

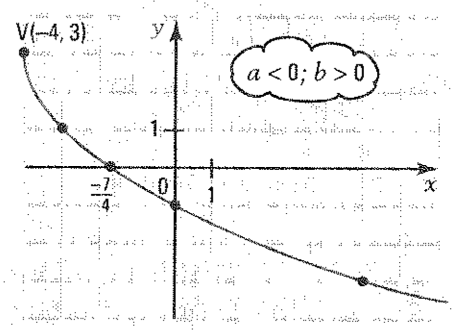
  If $ab > 0$, f is increasing over the domain.

  If $ab < 0$, f is decreasing over the domain.
- Extrema

  If $a > 0$, f has a minimum. min $f = k$.

  If $a < 0$, f has a maximum. max $f = k$.

  Ex.: Consider the function $f(x) = -2\sqrt{x+4} + 3$ ($a = -2$, $b = 1$, $h = -4$, $k = 3$).



- Vertex: V(–4, 3)
- dom $f = [-4, +\infty[$; ran $f =]-\infty, 3]$.
- Zero: $-2\sqrt{x+4} + 3 = 0$

$$\sqrt{x+4} = \frac{3}{2}$$

$$x + 4 = \frac{9}{4}$$

$$x = -\frac{7}{4}$$

- Initial value: $y = -1$
- Sign of f: $f(x) \geqslant 0$ over $\left[-4, -\frac{7}{4}\right]$; $f(x) \leqslant 0$ over $\left[-\frac{7}{4}, +\infty\right[$.
- Variation of f: f is decreasing, $\forall\, x \in$ dom f.
- max $f = 3$.

5. Write the rules of the square root functions in the form $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.

a) $y = -2\sqrt{4x+8} + 3$

$y = -4\sqrt{x+2} + 3$

b) $y = 2\sqrt{9x-36} + 4$

$y = 6\sqrt{x-4} + 4$

c) $y = -\frac{1}{2}\sqrt{18-9x} + 1$

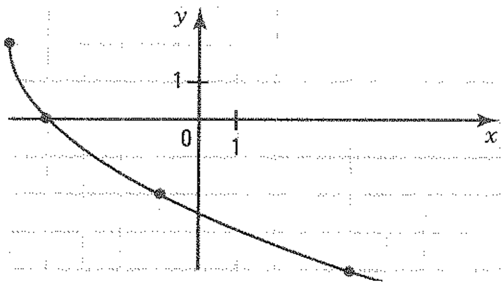
$y = -\frac{3}{2}\sqrt{-(x-2)} + 1$

d) $y = -\frac{3}{4}\sqrt{2-4x} + 7$

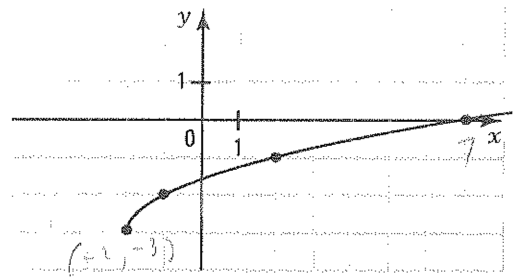
$y = -\frac{3}{2}\sqrt{-\left(x-\frac{1}{2}\right)} + 7$

6. Represent the following square root functions in the Cartesian plane.

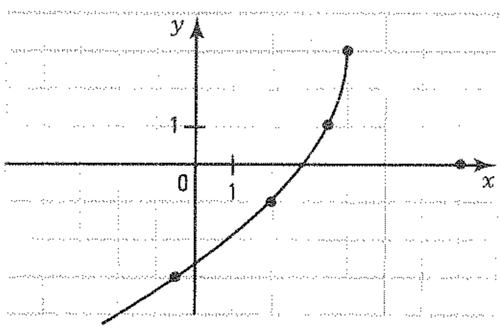
a) $y = -2\sqrt{x+5} + 2$



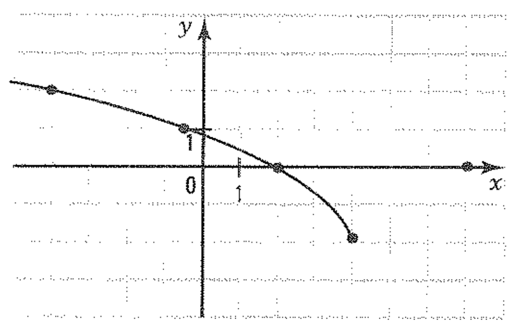
b) $y = \frac{1}{2}\sqrt{4x+8} - 3$



c) $y = -2\sqrt{-2(x-4)} + 3$



d) $y = \sqrt{-2(x-4)} - 2$



7. Consider the function $f(x) = 2\sqrt{x+4} - 2$.

a) Graph the function f.

b) Study the function f.

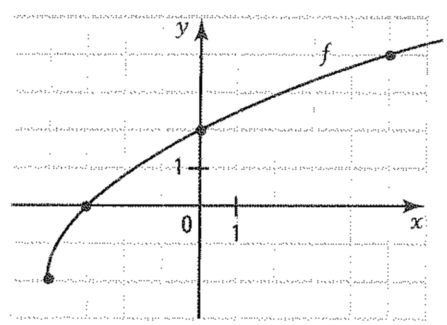
dom = $[-4, +\infty[$ ran = $[-2, +\infty[$

Zero: -3; initial value: 2

$f(x) \geq 0$ over $[-3, +\infty[$; $f(x) \leq 0$ over $[-4, -3]$

$f \nearrow, \forall x \in$ dom f

min $f = -2$



c) Using the graph of f, solve the inequality

1. $f(x) \geq 2$ $[0, +\infty[$

2. $f(x) \leq 4$ $[-4, 5]$

8. Determine the domain and range of the following functions.

a) $y = -2\sqrt{6 - 3x} + 4$

dom =]−∞, 2]; ran =]−∞, 4]

b) $y = 3\sqrt{4x + 2} - 1$

dom = $\left[-\frac{1}{2}, +\infty\right[$; ran = [−1, +∞[

9. Determine the zero and initial value of the following functions.

a) $y = -3\sqrt{6 - 4x} + 9$

zero: $-\frac{3}{4}$, i.v.: $-3\sqrt{6} + 9$

b) $y = 2\sqrt{4x - 1} - 1$

zero: $\frac{5}{16}$, i.v.: does not exist

c) $y = 2\sqrt{x - 5} + 4$

No zero, i.v.: does not exist

d) $y = -2\sqrt{3x + 1}$

zero: $-\frac{1}{3}$, i.v.: −2

10. Consider the absolute value function $f(x) = -2|6 - 2x| + 8$ and the square root function $g(x) = 3\sqrt{\frac{1}{2}(x + 4)} - 5$. Determine

a) $g \circ f(4) =$ *1*

b) $f \circ g(-2) =$ *−12*

11. Determine the interval over which each of these functions is positive.

a) $f(x) = 3\sqrt{x + 5} - 6$

f(x) ≥ 0 over [−1, +∞[

b) $f(x) = -2\sqrt{6 + 4x} + 4$

f(x) ≥ 0 over $\left[-\frac{3}{2}, -\frac{1}{2}\right]$

c) $f(x) = \frac{1}{2}\sqrt{4 - x} + 5$

f(x) ≥ 0 over]−∞, 4]

d) $f(x) = -3\sqrt{-2x + 8} - 1$

f(x) is never positive

12. Solve the following inequalities.

a) $-2\sqrt{x + 3} + 2 \geq 0$

S = [−3, −2]

b) $\sqrt{3x + 4} < -1$

S = ∅

c) $5\sqrt{2 - x} > 4$

S = $\left]-\infty, \frac{34}{25}\right[$

d) $\sqrt{\frac{1}{2}x + 8} > 0$

S =]−16, +∞[

13. Determine the interval over which each of these functions is increasing.

a) $f(x) = 3\sqrt{-2(x - 1)} + 5$

f is never increasing

b) $f(x) = -2\sqrt{-3(x + 4)}$

f ↗ over]−∞, −4]

14. Study each of the following functions and complete the following table.

	$f_1(x) = 3\sqrt{x-2} - 1$	$f_2(x) = -2\sqrt{\frac{1}{2}(x+4)} + 6$	$f_3(x) = \sqrt{2-x} + 1$	$f_4(x) = -2\sqrt{-x} + 4$
Domain	*[2, +∞[*	*[−4, +∞[*	*]−∞, 2]*	*]−∞, 0]*
Range	*[−1, +∞[*	*]−∞, 6]*	*[1, +∞[*	*]−∞, 4]*
Zero	$\frac{19}{9}$	*14*	***does not exist***	*−4*
Initial value	***does not exist***	$-2\sqrt{2} + 6$	$\sqrt{2} + 1$	*4*
Sign	***f(x) ≥ 0 over*** $\left[\frac{19}{9}, +\infty\right[$ ***f(x) < 0 over*** $\left[2, \frac{19}{9}\right[$	***f(x) ≥ 0 over [−4, 14]*** ***f(x) < 0 over]14, +∞[***	***f(x) ≥ 0 over]−∞, 2]*** ***f(x) < 0 never***	***f(x) ≥ 0 over [−4, 0]*** ***f(x) < 0 over]−∞, −4[***
Variation	***f ↗ over [2, +∞[*** ***f ↘ never***	***f ↗ never*** ***f ↘ over [−4, +∞[***	***f ↗ never*** ***f ↘ over]−∞, 2]***	***f ↗ over]−∞, 0]*** ***f ↘ never***
Extrema	***min = −1***	***max = 6***	***min = 1***	***max = 4***

ACTIVITY 6 Finding the rule of a square root function

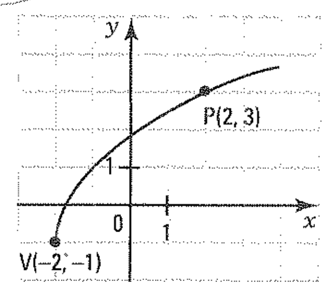
Any square root function can be written in the form $f(x) = a\sqrt{x-h} + k$ *or* $f(x) = a\sqrt{-(x-h)} + k$.

a) Consider the functions $f(x) = 3\sqrt{2x+4} - 5$ and $g(x) = 5\sqrt{-4x+8} - 1$.

Write the rule of each function in the form $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.

$f(x) = 3\sqrt{2(x+2)} - 5 = 3\sqrt{2} \cdot \sqrt{x+2} - 5$ **and** $g(x) = 5\sqrt{-4(x-2)} - 1 = 10\sqrt{-(x-2)} - 1$

b) We consider the function represented on the right.



1. Which of the two rules $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$ corresponds to the graph of this function?

   $y = a\sqrt{(x-h)} + k$

2. Identify h and k. $h = -2$, $k = -1$

3. Determine a knowing that the coordinates of the point P(2, 3) verify the rule of the function.

   $y = a\sqrt{x+2} - 1$

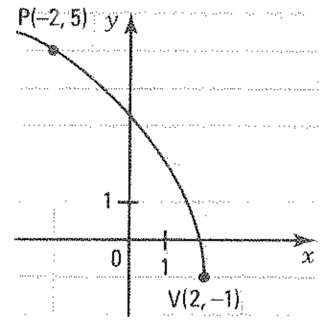
   $3 = a\sqrt{2+2} - 1$

   $4 = 2a$

   $a = 2$

4. What is the rule of the function? $y = 2\sqrt{x+2} - 1$

c) Consider the square root function whose graph has a vertex at V(2, −1) and passes through the point P(−2, 5).



1. Which of the two rules $y = a\sqrt{x - h} + k$ or $y = a\sqrt{-(x - h)} + k$ corresponds to the graph of this function?
   $y = a\sqrt{-(x - h)} + k$
2. Identify h and k. $h = 2, k = -1$
3. Determine a knowing that the coordinates of the point P(−2, 5) verify the rule of the function.
   $y = a\sqrt{-(x - 2)} - 1$; $5 = a\sqrt{-(-2 - 2)} - 1$; $6 = 2a$; $a = 3$
4. What is the rule of the function? $y = 3\sqrt{-(x - 2)} - 1$

d) What is the domain of a square root function if its rule is of the form

1. $f(x) = a\sqrt{x - h} + k$. dom $f = [h, +\infty[$
2. $f(x) = a\sqrt{-(x - h)} + k$. dom $f =]-\infty, h]$

FINDING THE RULE OF A SQUARE ROOT FUNCTION

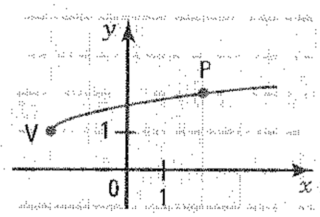
Any square root function can be written, depending on its domain, in the form:

$$f(x) = a\sqrt{x - h} + k \quad \text{or} \quad f(x) = a\sqrt{-(x - h)} + k$$

The vertex V and a point P are given.

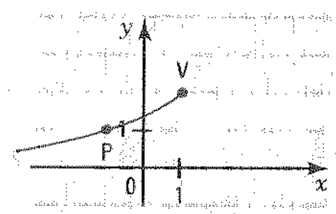
1. Determine the form of the rule, $y = a\sqrt{x - h} + k$ or $y = a\sqrt{-(x - h)} + k$.
2. Identify parameters h and k.
3. Determine a after replacing, in the rule, x and y by the coordinates of the point P.
4. Deduce the rule.

Ex.: **a)**



1. $y = a\sqrt{x - h} + k$
2. $y = a\sqrt{x + 2} + 1$
3. $2 = a\sqrt{2 + 2} + 1$
   $1 = 2a$
   $a = \frac{1}{2}$
4. rule: $y = \frac{1}{2}\sqrt{x + 2} + 1$

b)



1. $y = a\sqrt{-(x - h)} + k$
2. $y = a\sqrt{-(x - 1)} + 2$
3. $1 = a\sqrt{-(-1 - 1)} + 2$
   $-1 = a\sqrt{2}$
   $a = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
4. rule: $y = -\frac{\sqrt{2}}{2}\sqrt{-(x - 1)} + 2$
   or $y = -\frac{1}{2}\sqrt{-2(x - 1)} + 2$

15. Find the rule of each of the square root functions given its vertex V and a point P on its graph.

a) V(5, 3) and P(9, 3.5)
$y = \frac{1}{4}\sqrt{x - 5} + 3$

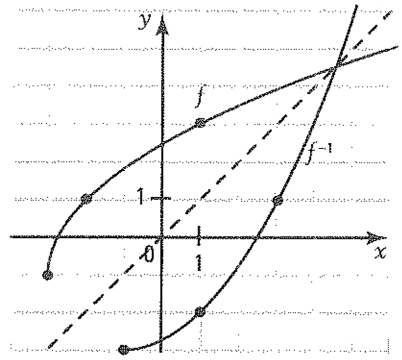
b) V(−2, −1) and P(−6, −4)
$y = -\frac{3}{2}\sqrt{-(x + 2)} - 1$

c) V(−2, 4) and P(23, 2)
$y = -\frac{2}{5}\sqrt{x + 2} + 4$

d) V(5, 3) and P(−13, 5)
$y = \frac{1}{3}\sqrt{-2(x - 5)} + 3$

ACTIVITY 7 Inverse of a square root function

Consider the function $f(x) = 2\sqrt{x + 3} - 1$.



a) In the same Cartesian plane,
1. graph the function f.
2. deduce the graph of f^{-1}.

b) Complete: The graphs of the function f and its inverse f^{-1} are symmetrical about the *y = x, bisector of the 1st quadrant*

c) Is the inverse f^{-1} a function? Justify your answer.
Yes, because any vertical line only intersects the graph of f^{-1} in at most one point.

d) 1. Determine
1) dom f ***[−3, +∞[*** 2) ran f ***[−1, +∞[*** 3) dom f^{-1} ***[−1, +∞[*** 4) ran f^{-1} ***[−3, +∞[***

2. Verify that
1) dom f^{-1} = ran f ***True*** 2) ran f^{-1} = dom f ***True***

e) Justify the steps in finding the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = 2\sqrt{x + 3} - 1$.

$y + 1 = 2\sqrt{x + 3}$ ***Add 1 to each side.***

$\frac{1}{2}(y + 1) = \sqrt{x + 3}$ ***Divide each side by 2.***

$\frac{1}{4}(y + 1)^2 = x + 3$ ***Square both sides.***

$\frac{1}{4}(y + 1)^2 - 3 = x$ ***Subtract 3 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse.
You get: $y = \frac{1}{4}(x + 1)^2 - 3$.

3. What restriction must be set on the variable x? Justify your answer.
$x \geq -1$ since dom f^{-1} = ran f = [−1, +∞[

The inverse of the square root function $y = 2\sqrt{x + 3} - 1$ is therefore the function
$y = \frac{1}{4}(x + 1)^2 - 3$ $(x \geq -1)$

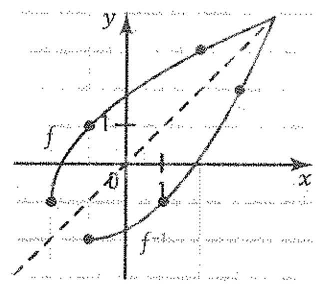
The graphic representation of the inverse corresponds to a semi-parabola.

INVERSE OF A SQUARE ROOT FUNCTION

The inverse of a square root function is a function whose graph is a semi-parabola.

Ex.: $f(x) = 2\sqrt{x+2} - 1$ has the inverse:

$$f^{-1}(x) = \frac{1}{4}(x+1)^2 - 2 \qquad (x \geqslant -1)$$



Note that dom f^{-1} = ran f = $[-1, +\infty[$.

The graphs of f and f^{-1} are symmetrical about the bisector of the 1st quadrant.

16. Determine the rule of the inverse of the following functions and indicate the domain of the inverse.

a) $y = 2\sqrt{x-1} + 7$

$y = \frac{1}{4}(x-7)^2 + 1$; dom = $[7, +\infty[$

b) $y = -3\sqrt{x+4} - 1$

$y = \frac{1}{9}(x+1)^2 - 4$; dom = $]-\infty, -1]$

c) $y = 4\sqrt{-(x+3)} - 2$

$y = -\frac{1}{16}(x+2)^2 - 3$; dom = $[-2, +\infty[$

d) $y = -2\sqrt{-(x-5)} + 4$

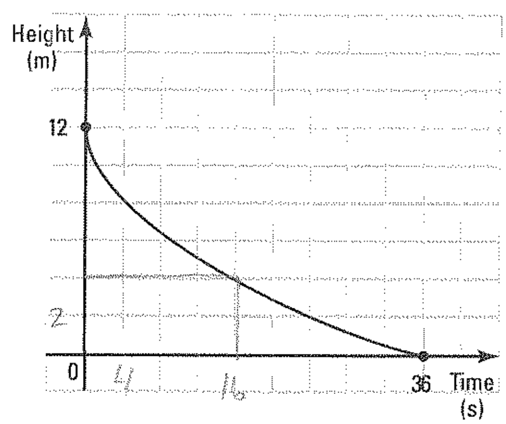
$y = -\frac{1}{4}(x-4)^2 + 5$; dom = $]-\infty, 4]$

17. At a water park, Raphael is getting ready to go down a slide.

The function f represented on the right gives Raphael's height h (in m) as a function of elapsed time t (in s) since his departure.

At what instant will he be at a height of 4 m?

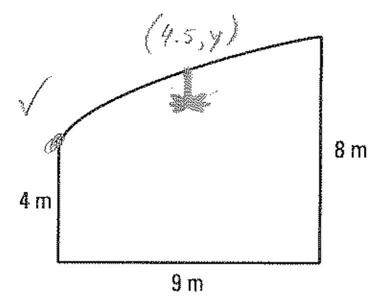
$h(t) = -2\sqrt{t} + 12$; after 16 seconds.



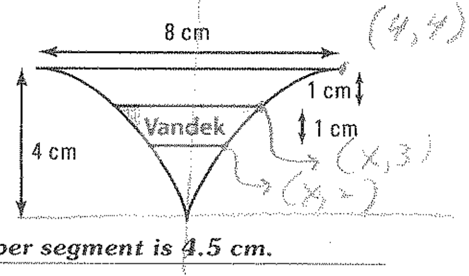
18. The lateral view of a solarium is represented by the graph on the right where the glass ceiling follows the curve of a square root function. A light is located at the centre of the room as indicated in the figure. Determine at what height the base of the light is located. (Round your answer to the nearest tenth.)

$V(0, 4)$; $y = a\sqrt{x} + 4$; $P(9, 8)$; $y = \frac{4}{3}\sqrt{x} + 4$.

When $x = 4.5$, the height is $y = 6.8$ m.



19. A company's logo is drawn using the graphs of two square root functions as illustrated in the figure on the right. The company's name is limited by two line segments.



a) What is the length of the upper segment?

Rule: $y = 2\sqrt{x}$. When $y = 3$, $x = 2.25$ cm. The length of the upper segment is 4.5 cm.

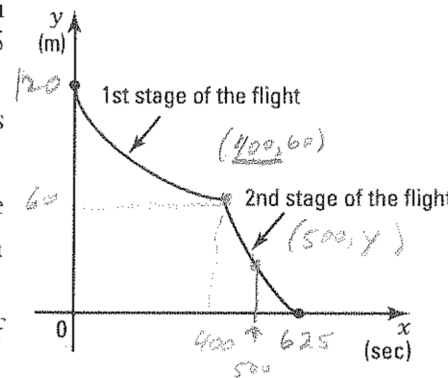
b) What is the length of the lower segment? ***2 cm***

20. The flight of a bird is observed from its takeoff at $t = 0$ from a 150 m high tower until it reaches the ground at $t = 625$ seconds.

The bird's flight is described by two square root functions represented in the figure on the right.

The 1st stage of its flight lasts 400 s and is described by the rule $y = -3\sqrt{t} + 120$ where t represents the time, in seconds, and y the height of the bird, in metres.

At the instant $t = 400$ s, the bird begins the second stage of its flight.



At what height will the bird be 500 s after the beginning of its flight?

$y = -4\sqrt{t - 400} + 60$. It will be at a height of 20 m.

$y = a\sqrt{x - h} + k$
$4 = a\sqrt{4 - 0} + 0$
$4 = 2a$
$a = 2$

a) $y = 2\sqrt{x}$
$3 = 2\sqrt{x}$
$9 = 4x$
$\frac{9}{4} = x$
$2 \times \frac{9}{4} = \frac{9}{2} =$ 4.5 cm

b) $2 = 2\sqrt{x}$
$1 = \sqrt{x}$
$x = 1$
2 cm

20.
$y = -3\sqrt{t} + 120$ (150?)
$y = -3\sqrt{400} + 120$
$y = 60$

$y = a\sqrt{x - h} + k$
$0 = a\sqrt{625 - 400} + 60$
$-60 = a(15)$
$-4 = a$
$y = -4\sqrt{x - 400} + 60$

$y = -4\sqrt{500 - 400} + 60$
$= -4\sqrt{100} + 60$
$=$ 20 m

3.5 Greatest integer function

ACTIVITY 1 Basic greatest integer function

a) The greatest integer of a real number x is represented by $[x]$.

1. What is the definition of the greatest integer of a real number x?

   The greatest integer of a real number x is equal to the greatest integer less than or equal to x.

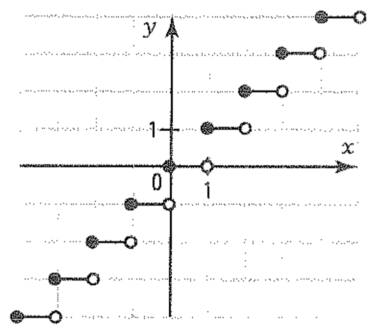
2. Calculate

   1) $[3.76]$ ***3*** 2) $[-1.25]$ ***-2***

3. In what interval is x located if

   1) $[x] = 2$ ***$x \in [2, 3[$*** 2) $[x] = -2$ ***$x \in [-2, -1[$***

b) 1. Graph the basic greatest integer function $f(x) = [x]$ in the Cartesian plane on the right.

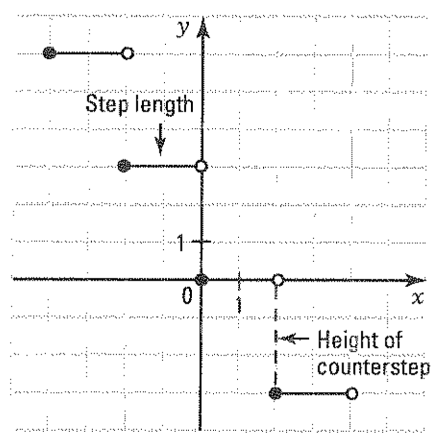


2. Determine

   1) dom f $\mathbb{R}$ 2) ran f $\mathbb{Z}$

ACTIVITY 2 General greatest integer function $f(x) = a[b(x - h)] + k$

The function $f(x) = -3\left[\frac{1}{2}(x + 4)\right] + 6$ is represented on the right.



a) Identify the parameters a, b, h and k.

$a = -3$, $b = \frac{1}{2}$, $h = -4$, $k = 6$

b) Verify that

1. each step has a length of $\frac{1}{|b|}$. ***$\frac{1}{|b|} = 2$***
2. The height of the counterstep is $|a|$. ***$|a| = |-3| = 3$***

c) Determine

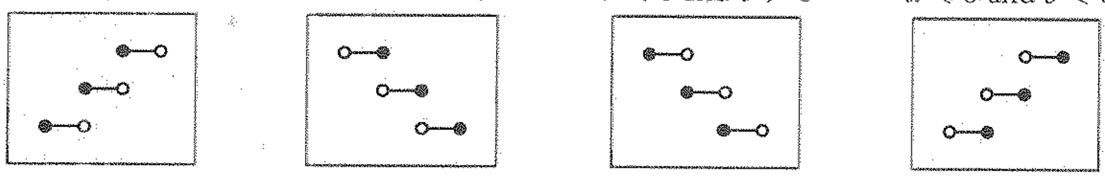
1. dom f $\mathbb{R}$ 2. ran f ***$\{y \mid y = -3m + 6, m \in \mathbb{Z}\}$***
3. the zeros of f. ***$[0, 2[$*** 4. the initial value of f. ***0***
5. the sign of f. ***$f(x) \geq 0$ over $]-\infty, 2[$; $f(x) \leq 0$ over $[2, +\infty[$***
6. the variation of f. ***$f \searrow$ over $\mathbb{R}$***

GREATEST INTEGER FUNCTION

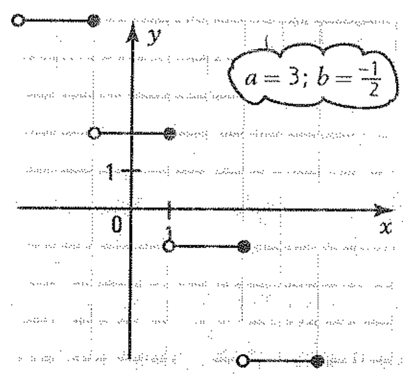
We consider the **greatest integer function** $f(x) = a[b(x - h)] + k$.

- The Cartesian graph is a step function.
- Each step has a length of $\frac{1}{|b|}$.
  - If $b > 0$, the steps are closed on the left and open on the right (●—○).
  - If $b < 0$, the steps are open on the left and closed on the right (○—●).
- The height of the counterstep is $|a|$.
- $\text{dom } f = \mathbb{R}$, $\text{ran } f = \{y \mid y = am + k, m \in \mathbb{Z}\}$
- – If $ab > 0$, the function is increasing.
  – If $ab < 0$, the function is decreasing.
- The function f has zeros if and only if k is a multiple of a.
- The signs of a and b help us distinguish 4 cases:

$a > 0$ and $b > 0$ | $a > 0$ and $b < 0$ | $a < 0$ and $b > 0$ | $a < 0$ and $b < 0$



Ex.: Given the function $f(x) = 3\left[-\frac{1}{2}(x - 1)\right] + 2$.



We have: $a = 3$; $b = \frac{-1}{2}$; $h = 1$ and $k = 2$.

- The length of a step is $\frac{1}{|b|} = 2$.
- The height of a counterstep is $|a| = 3$.
- $\text{dom } f = \mathbb{R}$
- $\text{ran } f = \{y \mid y = 3m + 2, m \in \mathbb{Z}\}$
- zeros of f: f has no zeros, since k is not a multiple of a.
- initial value of f: 2.
- $f(x) > 0$ if $x \leq 1$; $f(x) < 0$ if $x > 1$
- f is decreasing over $\mathbb{R}$ since $ab < 0$.
- f has no extrema.

1. Determine the domain and range of the following functions.

a) $y = 4\left[\frac{1}{3}(x - 5)\right] - 2$

dom = ℝ; ran = {y | y = 4m − 2, m ∈ ℤ}

b) $y = -2[4(x + 1)] + 4$

dom = ℝ; ran = {y | y = −2m + 4, m ∈ ℤ}

2. Determine the zeros of the following functions.

a) $y = 2\left[\frac{1}{4}(x + 1)\right] - 6$

[11, 15[

b) $y = -3[2(x - 4)] - 12$

[2, 2.5[

c) $y = 4[2x] + 2$

No zero

d) $y = -5[x - 8]$

[8, 9[

3. Determine the initial value of the function $f(x) = -3\left[\frac{1}{5}(x-9)\right] + 10$ *16*

4. Determine over what interval the function $f(x) = 3\left[\frac{1}{4}(x-1)\right] + 6$ is positive. *[-7, +∞[*

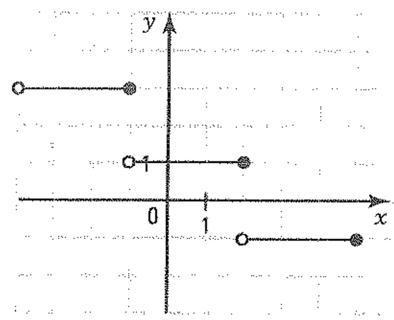
5. Determine over what interval the function $f(x) = 5[x-3] + 1$ is strictly negative. *]-∞, 3[*

6. Determine over what interval the function $f(x) = 3\left[\frac{1}{2}(x-7)\right] + 6$ is increasing. *f ↗ over ℝ*

7. Consider the functions $f(x) = 2\left[\frac{1}{4}(x-1)\right] + 2$ and $g(x) = -3\sqrt{x+5} + 4$.

Determine $g \circ f(7) =$ *-5*

8. Determine the rule of the greatest integer function represented on the right.



We choose (h, k) = (2, 1)

o—● implies that $b < 0$, $b = -\frac{1}{3}$

f ↘ implies that $a > 0$, $a = 2$

Rule: $y = 2\left[-\frac{1}{3}(x-2)\right] + 1$

9. A salesman in a store receives a weekly base salary of \$300 plus a commission of \$40 for every 10 items he sells during that week.

a) Find the rule of the function which gives the salesman's salary y as a function of the number of items sold x. $y = 40\left[\frac{x}{10}\right] + 300$

b) What is this salesman's salary if he sold 84 items this week? ***\$620***

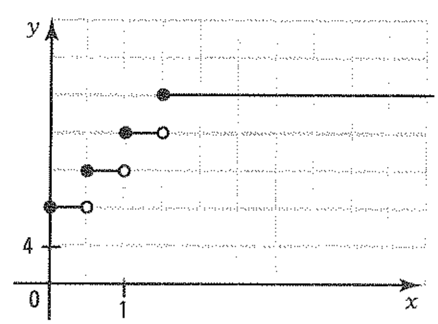
c) In what interval is the number of items sold if the salesman's salary is \$500?

In the interval [50, 60[

d) Can this salesman earn a salary of \$450? Justify your answer.

No, the equation $40\left[\frac{x}{10}\right] + 300 = 450$ ***has no solution since*** $\left[\frac{x}{10}\right] \neq 3.75$.

10. The cost of parking in a lot is \$8 for a duration of less than 30 min. Afterward, the cost increases by \$4 for every 30 minutes or part thereof. The maximum cost is \$20 per day.



a) What is the rule of the function which gives the cost y (in \$) as a function of the parking duration x (in hours).

$y = 4[2x] + 8$

b) Represent this situation in the Cartesian plane on the right.

c) What is the cost for a parking duration of 1 h 40 min? ***\$20***

d) In what interval is the parking duration if the cost is \$12?

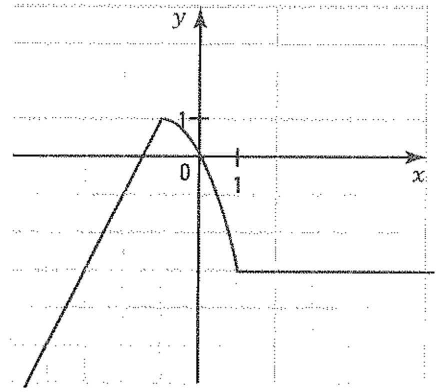
In the interval $\left[\frac{1}{2}, 1\right[$

3.6 Piecewise function

ACTIVITY 1 Graph of a piecewise function

A function f is defined by three different rules, depending on the interval over which x is located.

- Over the interval $]-\infty, -1]$, the function f is defined by the rule $f(x) = 2x + 3$.
- Over the interval $]-1, 1]$, the function f is defined by the rule $f(x) = -(x + 1)^2 + 1$.
- Over the interval $]1, +\infty]$, the function f is defined by the rule $f(x) = -3$.



a) Represent, in the Cartesian plane on the right, the function f.

b) Determine

1. $f(-2) =$ *-1* 2. $f(0) =$ *0* 3. $f(5) =$ *-3*

c) Find

1. dom $f =$ $\mathbb{R}$ 2. ran $f =$ *]-∞, 1]*

d) Find

1. the zero of f. *0*, −1.5 2. the initial value of f. *0*

e) Determine over what interval the function is positive. $\left[-\frac{3}{2}, 0\right]$

f) Determine over what interval the function is

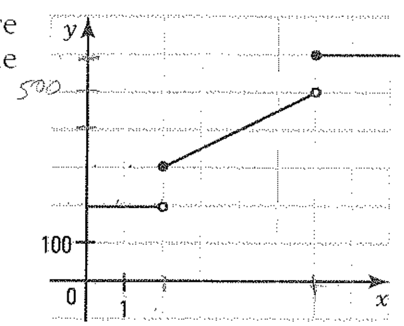
1. strictly increasing. *]-∞, -1]* 2. strictly decreasing. *[-1, 1]*
3. constant. *[1, +∞[*

g) Does the function f have any extrema? If yes, what? *Yes, a maximum; max f = 1*

ACTIVITY 2 An employee's salary

The weekly salary $f(x)$ of an employee in an electronic games store is calculated, according to the number x of games sold, using the following rule:

$$f(x) = \begin{cases} 200 & \text{if } x < 2 \\ 50x + 200 & \text{if } 2 \leq x < 6 \\ 600 & \text{if } x \geq 6 \end{cases}$$



a) What is the salary of an employee who sells

1. 2 games? *300 $* 2. 4 games? *400 $* 3. 12 games? *600 $*

b) Determine the number of games sold by an employee whose salary is

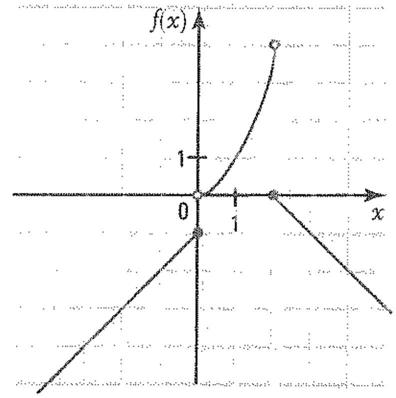
1. $200. *0 or 1 game sold* 2. $450. *5 games* 3. $600. *6 games or more*

PIECEWISE FUNCTIONS

A **piecewise function** is a function whose rule differs depending on the interval over which the variable x is located.

Ex.: Consider the following function.

$$f(x) = \begin{cases} x - 1 & \text{if } x \leqslant 0 \\ x^2 & \text{if } 0 < x < 2 \\ -x + 2 & \text{if } x \geqslant 2 \end{cases}$$



The graph of this function is represented in the Cartesian plane on the right.

dom $f = \mathbb{R}$, ran $f =]-\infty, 4[$

When we evaluate this function for a given value of the variable x, we find in which interval this value belongs to and we use the rule of the function defined over this interval.

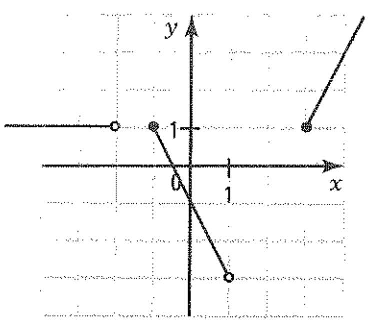
Thus, $f(1.5) = (1.5)^2 = 2.25$; $f(3) = -(3) + 2 = -1$.

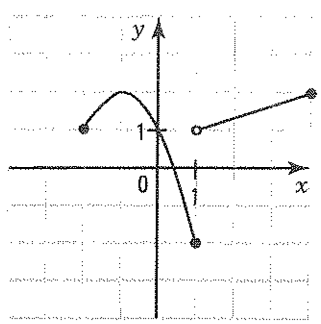
1. Graph the following functions.

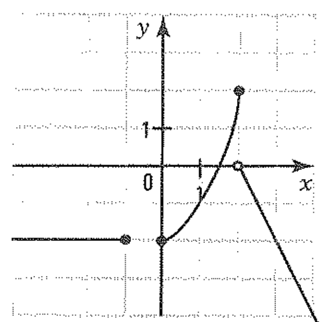
a) $f_1(x) = \begin{cases} 1 & \text{if } x < -2 \\ -2x - 1 & \text{if } -1 \leqslant x < 1 \\ 2x - 5 & \text{if } x \geqslant 3 \end{cases}$

b) $f_2(x) = \begin{cases} -(x+1)^2 + 2 & \text{if } -2 \leqslant x \leqslant 1 \\ \frac{1}{3}x + \frac{2}{3} & \text{if } 1 < x \leqslant 4 \end{cases}$

c) $f_3(x) = \begin{cases} -2 & \text{if } x \leqslant -1 \\ x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ -2x + 4 & \text{if } x > 2 \end{cases}$







2. For each of the piecewise functions given in number 1, find

a) the domain and range.

dom $f_1 =]-\infty, -2[\cup [-1, 1[\cup [3, +\infty[$; ***ran*** $f_1 =]-3, +\infty[$

dom $f_2 = [-2, 4]$; ***ran*** $f_2 = [-2, 2]$

dom $f_3 =]-\infty, -1] \cup [0, +\infty[$; ***ran*** $f_3 =]-\infty, 2]$

b) the image of 2.

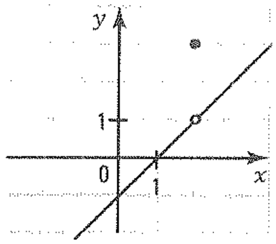
$f_1(2)$: ***does not exist;*** $f_2(2)$: $\frac{4}{3}$; $f_3(2)$: 2

c) the initial value.

$y_1 = -1$; $y_2 = 1$; $y_3 = -2$

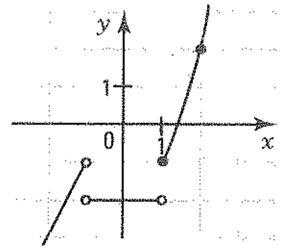
3. Graph the following functions and determine their domain.

a) $f(x) = \begin{cases} x - 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$



dom f = ℝ

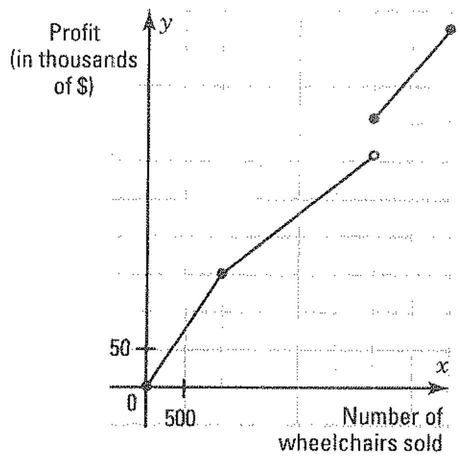
b) $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 1 \\ x^2 - 2 & \text{if } x \geq 1 \end{cases}$



dom f = ℝ \ {−1}

4. The Kandev company sells wheelchairs to residences for the elderly. The function f which gives the annual net profit y (in thousands of dollars) as a function of the number x of wheelchairs sold is given by the rule:

$$f(x) = \begin{cases} 0.15x & 0 \leq x \leq 1000 \\ 0.08x + 70 & 1000 < x < 3000 \\ 0.12x & 3000 \leq x \leq 4000 \end{cases}$$



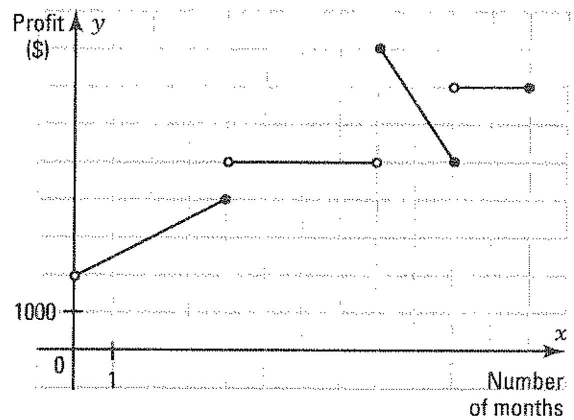
a) If the maximum number of wheelchairs sold per year is 4000, draw the graph of this function.

b) Find dom f. ***[0, 4000]***

c) What is the profit made from selling 2500 wheelchairs?
$270

d) Over what interval is the rate of change the greatest? ***[0, 1000]***

5. The piecewise function f represented on the right gives a company's accumulated profit $f(x)$ as a function of the number x of elapsed months.



a) What is the company's accumulated profit after

1. 2 months? ***$3000***   2. 4 months? ***$4000***
3. 6 months? ***$5000***   4. 11 months? ***$7000***

b) Determine the number of elapsed months if the company's accumulated profit is

1. $3000. ***2 months***
2. $6500. ***9 months***

c) Determine the rule of the function f.

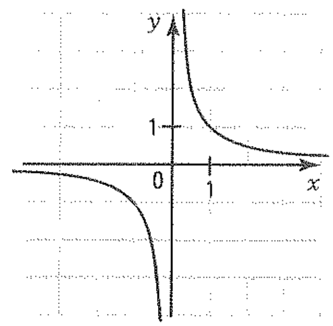
$$f(x) = \begin{cases} 500x + 2000 & \text{if } 0 < x \leq 4 \\ 5000 & \text{if } 4 < x < 8 \\ -1500x + 20\,000 & \text{if } 8 \leq x \leq 10 \\ 7000 & \text{if } 10 < x \leq 12 \end{cases}$$

d) Over what interval is the function f

1. strictly increasing? ***]0, 4]***
2. strictly decreasing? ***[8, 10]***
3. constant? ***]4,8[or]10,12]***

3.7 Rational function

ACTIVITY 1 Basic rational function



Consider the function f defined by the rule: $f(x) = \frac{1}{x}$.

a) What restriction must be imposed on the variable x?

x must be a non-zero real number.

b) Complete the following table.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4		4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

c) Indicate what number the variable y approaches as

1. the variable x takes positive values that are bigger and bigger. ***0***
2. the variable x takes negative values that are smaller and smaller. ***0***

d) Indicate the behaviour of the variable y as

1. the variable x takes positive values closer and closer to zero.

   The variable y takes bigger and bigger positive values.
2. the variable x takes negative values closer and closer to zero.

   The variable y takes smaller and smaller negative values.

e) Graph the function in the Cartesian plane above.

f) Observe the branch of the hyperbola located in the 1st quadrant.

1. When x takes positive values that are bigger and bigger, the branch gets closer and closer to the x-axis without ever touching it. *We say that the x-axis is a **horizontal asymptote** to the curve.* What is the equation of this asymptote? ***y = 0***
2. When x takes positive values closer and closer to zero, the branch gets closer and closer to the y-axis without ever touching it. *We say that the y-axis is a **vertical asymptote** to the curve.* What is the equation of this asymptote? ***x = 0***

g) Observe the branch of the hyperbola located in the 3rd quadrant.

1. Do we observe a horizontal asymptote? If yes, what is its equation?

   Yes; y = 0
2. Do we observe a vertical asymptote? If yes, what is its equation?

   Yes; x = 0

The represented curve is called a **hyperbola**. This hyperbola consists of two branches. Place a random point M(x, y) on a branch and verify that the point M′($-x$, $-y$) is located on the other branch. The origin O, mid-point of the segment MM′, is therefore called the **symmetric centre** of the hyperbola.

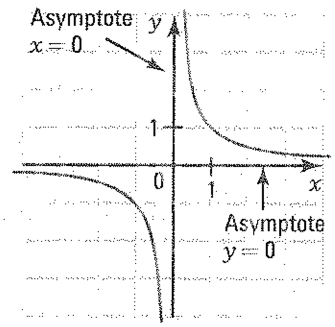
h) Determine

1. dom $f =$ $\mathbb{R}^*$
2. ran $f =$ $\mathbb{R}^*$
3. the zero of f. ***does not exist***
4. the initial value of f. ***does not exist***
5. the sign of f. ***$f(x) \geq 0$ over $\mathbb{R}^*_+$; $f(x) < 0$ over $\mathbb{R}^*_-$.***
6. the variation of f. ***$f \searrow$ over $\mathbb{R}^*$; f is never increasing.***
7. the extrema of f (if it exists). ***does not exist***

BASIC RATIONAL FUNCTION

- Consider the rational function defined by the rule:

$$f(x) = \frac{1}{x}$$



This function is called the **basic rational function**.

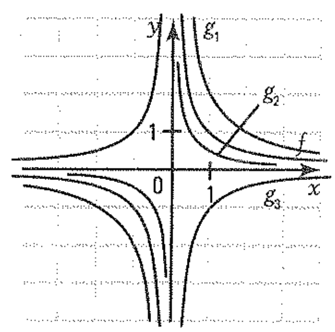
- We have:
  - dom $f = \mathbb{R}^*$, ran $f = \mathbb{R}^*$.
  - f has no zeros.
  - f is decreasing over $\mathbb{R}^*$.
  - The represented curve is called a **hyperbola**. This hyperbola consists of two branches.
  - The origin 0 is the **symmetrical centre** of the hyperbola.
  - The hyperbola has two **asymptotes**: the x-axis and the y-axis.

ACTIVITY 2 Role of the parameters a, b, h and k

The basic rational function $f(x) = \frac{1}{x}$ can be transformed into a rational function with the rule

$$g(x) = \frac{a}{b(x-h)} + k \quad \text{(standard form)}$$

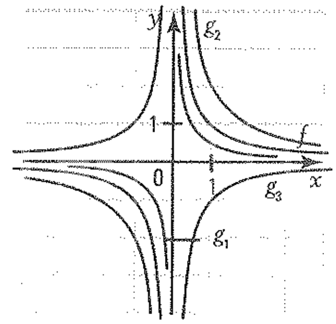
a) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{a}{x}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{2}{x}$, $g_2(x) = \frac{0.5}{x}$ and $g_3(x) = \frac{-1}{x}$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection about the x-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{a}{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.

b) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{bx}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{2x}$, $g_2(x) = \frac{1}{0.5x}$ and $g_3(x) = \frac{1}{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection about the y-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$.
5. Compare the graphs of f and g in each of the following cases and justify your answer.

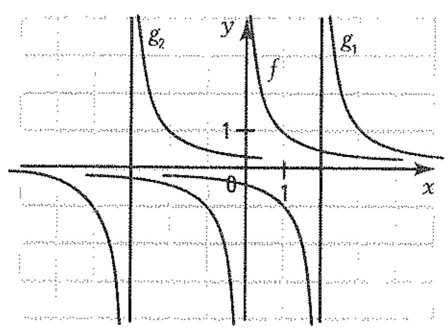
   $f(x) = \frac{2}{x}$ and $g(x) = \frac{1}{0.5x}$: ***They are the same. In fact,*** $\frac{1}{0.5x} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$.

   $f(x) = \frac{0.5}{x}$ and $g(x) = \frac{1}{2x}$: ***They are the same. In fact,*** $\frac{0.5}{x} = \frac{\frac{1}{2}}{x} = \frac{1}{2x}$.

   $f(x) = \frac{-1}{x}$ and $g(x) = \frac{1}{-x}$: ***They are the same. In fact,*** $\frac{-1}{x} = \frac{1}{-x}$.

   The reflection about the x-axis and the reflection about the y-axis have the same effect on the basic rational function.

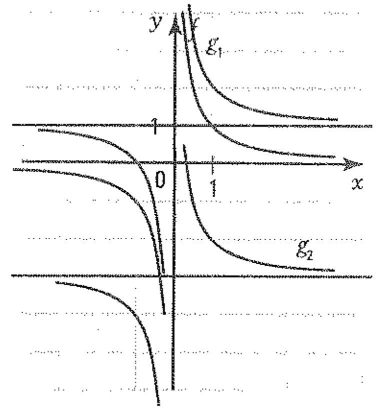
c) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x - h}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x - 2}$, $g_2(x) = \frac{1}{x + 3}$ and explain how to deduce the graph of g from the graph of f when

1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. What is the equation of the vertical asymptote of the function $g(x) = \frac{1}{x-h}$? ***$x = h$***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x - h}$ by the transformation $(x, y) \rightarrow$ ***$(x + h, y)$***.

d) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x} + k$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x} + 1$, $g_2(x) = \frac{1}{x} - 3$ and explain how to deduce the graph of g from the graph of f when

1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. What is the equation of the horizontal asymptote of the function $g(x) = \frac{1}{x} + k$? ***y = k.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x} + k$ by the transformation $(x, y) \rightarrow$ ***(x, y + k)***.

RATIONAL FUNCTION – STANDARD FORM

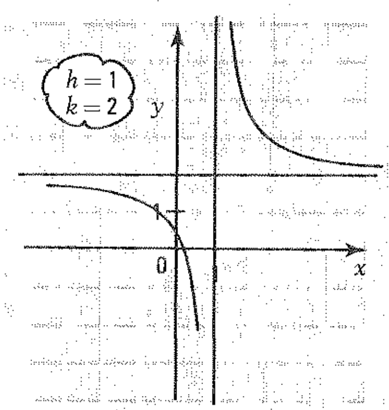
- The graph of the function

$$f(x) = \frac{a}{b(x - h)} + k$$

is deduced from the graph of the basic rational function $y = \frac{1}{x}$ by the transformation

$$(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$$

- This hyperbola has two asymptotes, the **vertical** asymptote with equation $x = h$ and the **horizontal** asymptote with equation $y = k$.
- The point (h, k) is the **symmetrical centre** of the hyperbola.



Ex.: To graph the hyperbola $y = \frac{3}{2(x - 1)} + 2$,

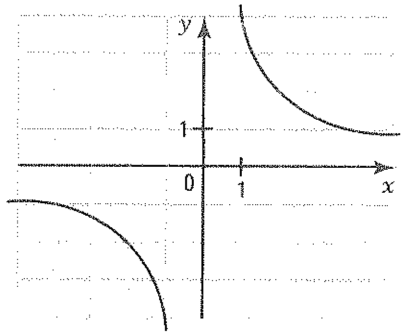
1. we draw the asymptotes.
   – vertical asymptote: $x = 1$.
   – horizontal asymptote: $y = 2$.
2. we complete a table of values.

x	–2	–1	0	1	2	3	4
y	1.5	1.25	0.5		3.5	2.75	2.5

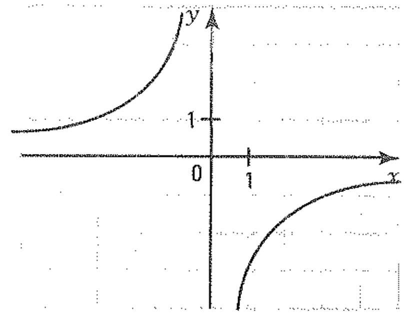
3. we draw the hyperbola using the symmetrical centre (h, k).

1. Graph the following rational functions.

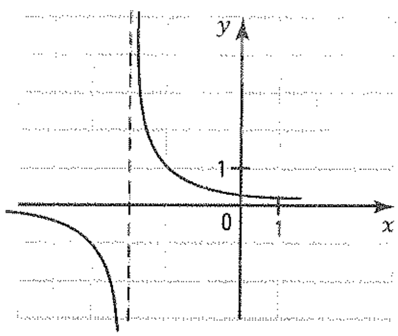
a) $f(x) = \frac{4}{x}$.



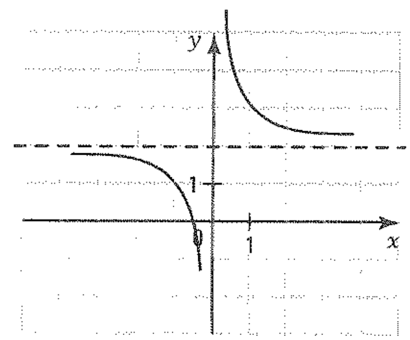
b) $f(x) = -\frac{3}{x}$.



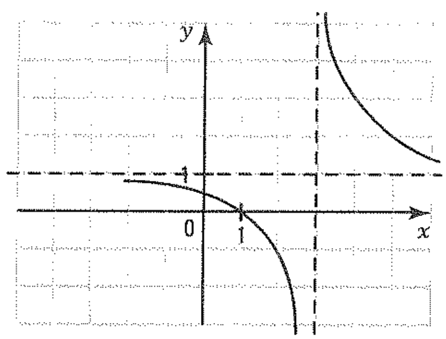
c) $f(x) = \frac{1}{x+3}$.



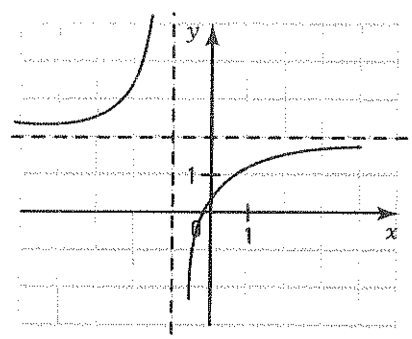
d) $f(x) = \frac{1}{x} + 2$.



e) $f(x) = \frac{2}{x-3} + 1$.

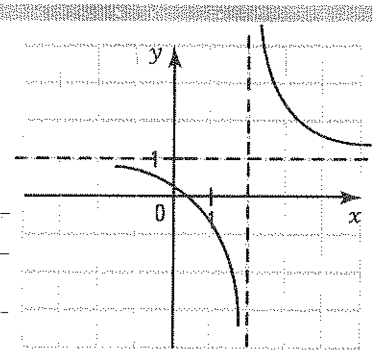


f) $f(x) = \frac{3}{-2(x+1)} + 2$.



ACTIVITY 3 Study of a rational function

Consider the function f defined by the rule $y = \frac{3}{2(x-2)} + 1$.



a) Graph the function in the Cartesian plane on the right.

b) Determine

1. dom f = ***$\mathbb{R}\setminus\{2\}$***
2. ran f = ***$\mathbb{R}\setminus\{1\}$***
3. the zero of f (if it exists). ***0.5***
4. the initial value of f. ***0.25***
5. the sign of f. ***$f(x) \geq 0$ over $]-\infty, \frac{1}{2}] \cup]2, +\infty[$; $f(x) \leq 0$ over $[\frac{1}{2}, 2[$***
6. the variation of f. ***$f \searrow$ over $\mathbb{R}\setminus\{2\}$; $f \nearrow$ never***
7. the extrema of f. ***does not exist***

ACTIVITY 4 Finding the zero of a rational function

a) Consider the function defined by the rule: $y = \frac{-3}{4(x-2)} + 5$.

Justify the steps which enable you to find the zero of this function.

$\frac{-3}{4(x-2)} + 5 = 0$ *Replace y by zero.*

$\frac{-3}{4(x-2)} = -5$ *Subtract 5 from each side.*

$-20(x-2) = -3$ *The cross products are equal.*

$x - 2 = \frac{3}{20}$ *Divide each side by -20.*

$x = \frac{43}{20}$ *Add 2 to each side.*

b) Under what condition does the zero of a rational function defined by the rule $y = \frac{a}{b(x-h)} + k$ exist? *If $k \neq 0$*

STUDY OF A RATIONAL FUNCTION

Consider the **rational** function f defined by the rule:

$$f(x) = \frac{a}{b(x-h)} + k \quad \text{(standard form)}$$

- dom $f = \mathbb{R}\backslash\{h\}$; ran $f = \mathbb{R}\backslash\{k\}$
- The **zero** of f exists if $k \neq 0$, and the **initial value** of f exists if $h \neq 0$.
- To study the **sign** of f,
  - we find the zero (if it exists);
  - we establish the sign of f using a sketch of the graph.
- **Variation**
  - If $ab > 0$, f is decreasing over the domain.
  - If $ab < 0$, f is increasing over the domain.
- The rational function has no **extrema**.

2. Determine the domain and range of the following functions.

a) $y = \frac{-2}{4(x+5)} - 7$

dom = $\mathbb{R}\backslash\{-5\}$; ran = $\mathbb{R}\backslash\{-7\}$

b) $y = \frac{3}{2(x-1)} + 4$

dom = $\mathbb{R}\backslash\{1\}$; ran = $\mathbb{R}\backslash\{4\}$

3. Determine the zero and initial value of the following functions.

a) $y = \frac{3}{x-5} + 4$

Zero: $\frac{17}{4}$; i.v.: $\frac{17}{5}$

b) $y = \frac{-2}{3(x+1)}$

Zero: none; i.v.: $-\frac{2}{3}$

c) $y = \frac{-5}{4x} + 10$

Zero: $\frac{1}{8}$; i.v.: none

4. Determine the interval over which the function $f(x) = \frac{-4}{5(x-1)} + 3$ is positive. $]-\infty, 1[\cup [\frac{19}{15}, +\infty[$

5. Determine the interval over which the function $f(x) = \frac{3}{2(x+2)} - 1$ is strictly positive. $\left]-2, \frac{-1}{2}\right[$

6. Study the variation of the function $f(x) = \frac{-2}{5(x-1)} + 4$. *f ↗ over* $\mathbb{R}\setminus\{1\}$

7. Consider the functions $f(x) = -2|-x+4| + 5$, $g(x) = 3\sqrt{x-3} + 2$, $h(x) = \frac{6}{5(x-1)} + \frac{22}{5}$ and $i(x) = 3\left[\frac{1}{4}(x-2)\right] + 1$. Determine $f \circ g \circ h \circ i\,(1) =$ **3**

8. Given $f(x) = \frac{3}{2(x-4)} + 1$ and $g(x) = 3x - 1$. Determine, in standard form, the rule of the function $f \circ g$. $f \circ g(x) = f(3x-2) = \frac{3}{2(3x-6)} + 1 = \frac{1}{2(x-2)} + 1.$

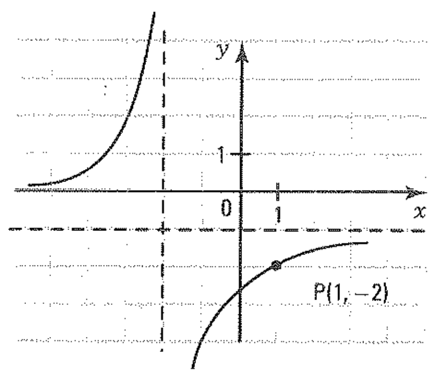
ACTIVITY 5 Finding the rule of a rational function

Any rule of a rational function can be written in the form $y = \frac{a}{x-h} + k$.

a) Consider the function $y = \frac{-3}{6(x-2)} + 1$. Write the rule of this function in the form $y = \frac{a}{x-h} + k$. $y = \frac{-0.5}{x-2} + 1$

b) Consider a rational function whose graph passes through the point P(1, –2).

1. Identify h and k. $h = -2, k = -1$
2. Determine a knowing that the coordinates of the point P(1, –2) verify the rule of the function.
   We have: $y = \frac{a}{x+2} - 1; -2 = \frac{a}{1+2} - 1; -1 = \frac{a}{3}; a = -3.$



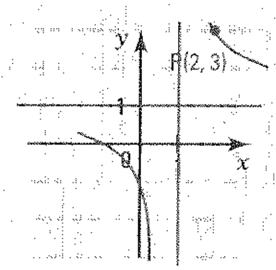
3. What is the rule of the function? $y = \frac{-3}{x+2} - 1$

FINDING THE RULE OF A RATIONAL FUNCTION

Any rule of a rational function can be written in the form

$$y = \frac{a}{x-h} + k$$

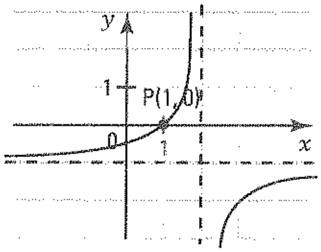
The asymptotes and a point are known.



1. Identify the parameters h and k. — 1. $h = 1$ and $k = 1$; $y = \frac{a}{x-1} + 1$
2. Find a after replacing, in the rule, x and y by the coordinates of the given point P. — 2. $3 = \frac{a}{2-1} + 1$; $a = 2$
3. Deduce the rule. — 3. $y = \frac{2}{x-1} + 1$

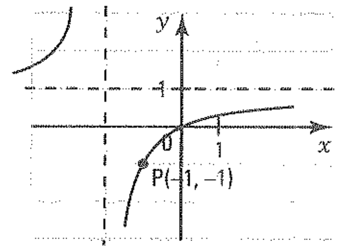
9. Find the rule of the following rational functions.

a)



$y = \frac{-1}{x-2} - 1$

b)



$y = \frac{-2}{x+2} + 1$

ACTIVITY 6 Inverse of a rational function – Standard form

a) Consider the rational function f defined by the rule: $y = \frac{-2}{3(x+1)} - 5$.

Justify the steps which enable you to determine the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = \frac{-2}{3(x+1)} - 5$

$y + 5 = \frac{-2}{3(x+1)}$ — ***Add 5 to each side.***

$3(x+1) = \frac{-2}{y+5}$ — ***Switch the extremes.***

$x + 1 = \frac{-2}{3(y+5)}$ — ***Divide each side by 3.***

$x = \frac{-2}{3(y+5)} - 1$ — ***Subtract 1 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse. We get:

$y = \frac{-2}{3(x+5)} - 1$

b) Complete: The inverse of a rational function is a ***rational function.***

c) 1. Determine

1) dom f = $\mathbb{R}\setminus\{-1\}$ 2) ran f = $\mathbb{R}\setminus\{-5\}$

3) dom f^{-1} = $\mathbb{R}\setminus\{-5\}$ 4) ran f^{-1} = $\mathbb{R}\setminus\{-1\}$

2. Verify that dom f^{-1} = ran f and that ran f^{-1} = dom f.

INVERSE OF A RATIONAL FUNCTION

The **inverse** of a rational function is a rational function.

Ex.: Given the rational function defined by the rule $y = \frac{-2}{3(x+1)} - 5$.

The inverse f^{-1} is a rational function defined by the rule $y = \frac{-2}{3(x+5)} - 1$.

(See activity 6 for finding the rule of f^{-1})

Note that $\text{dom } f = \text{ran } f^{-1} = \mathbb{R}\backslash\{1\}$ and that $\text{ran } f = \text{dom } f^{-1} = \mathbb{R}\backslash\{5\}$

10. Determine the inverse of the following rational functions.

a) $y = \frac{3}{x+5} - 1$ *$y = \frac{3}{x+1} - 5$*

b) $y = \frac{-1}{2(x-4)} + 3$ *$y = \frac{-1}{2(x-3)} + 4$*

11. A train travels a distance of 240 km. We consider the function f which gives the duration t (in h) of the trip as a function of the train's speed v (in km/h).

v (km/h)	40	60	80	120	160
t (h)	*6*	*4*	*3*	*2*	*1.5*

a) Complete the table of values on the right.

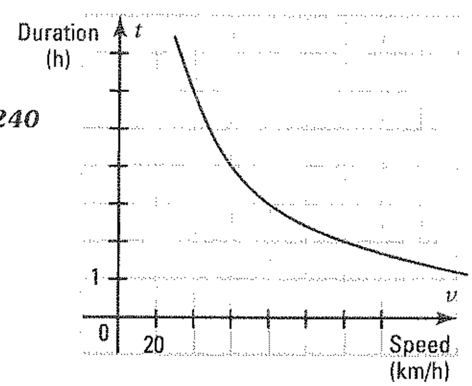
b) Is the rate of change of the function f constant? *No*

c) Verify that the product of the variables vt is constant. *$vt = 240$*

We say that the duration of the trip is **inversely proportional** to the speed or that the speed is inversely proportional to the duration.

d) What is the rule of the function? *$t = \frac{240}{v}$*

e) Graph the function f in the Cartesian plane.



f) Determine

1. dom f. *]0, +∞[*
2. ran f. *]0, +∞[*

g) When one variable increases, does the other variable increase or decrease? *It decreases.*

h) Is the function f increasing or decreasing? Justify your answer.

Decreasing, since the duration decreases as the speed increases.

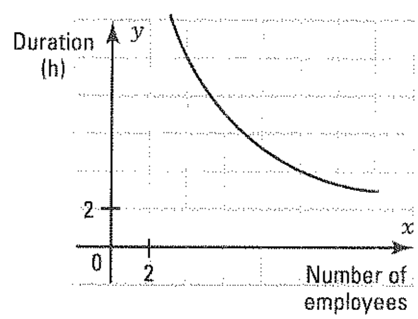
12. Renovations to a home require a total of 40 h of work for one employee. Consider the function f which gives the duration y (in h) of work per employee as a function of the number of employees x hired to do the renovations.

a) Complete the following table of values.

x	1	2	4	5	8	10
y	*40*	*20*	*10*	*8*	*5*	*4*

b) What is the rule of function f? *$y = \frac{40}{x}$*

c) Graph the function f in the Cartesian plane.



d) Is the function f increasing or decreasing? *Decreasing*

ACTIVITY 7 Rational function – General form

Consider the rational function defined by the rule $y = \frac{3}{2(x-5)} + 4$ (standard form).

a) Justify the steps which enable you to write the rule of this function in the form $y = \frac{ax+b}{cx+d}$.

$y = \frac{3}{2(x-5)} + 4 = \frac{3}{2(x-5)} + \frac{8(x-5)}{2(x-5)}$ ***Finding a common denominator***

$= \frac{3+8(x-5)}{2(x-5)}$ ***Addition of the 2 fractions;*** $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

$= \frac{8x-37}{2x-10}$ ***Simplification***

The form $y = \frac{ax+b}{cx+d}$ *is called the* ***general form*** *of a rational function.*

b) 1. Identify the parameters h and k of the standard form. ***h = 5; k = 4***

2. Identify the parameters a, b, c and d of the general form. ***a = 8, b = −37, c = 2, d = −10***

3. Verify that the vertical asymptote has the equation $x = -\frac{d}{c}$. ***x = h = 2 and x = −*** $\frac{d}{c}$ ***= 2***

4. Verify that the horizontal asymptote has the equation $y = \frac{a}{c}$. ***y = k = 4 and y =*** $\frac{a}{c}$ ***= 4***

c) Consider the rational function $y = \frac{5x-3}{2x+4}$ (general form).

To obtain the standard form from the general form $y = \frac{A(x)}{B(x)}$ where $A(x) = 5x - 3$ and $B(x) = 2x + 4$, we proceed in the following manner:

1° Determine the quotient Q(x) and the remainder R(x) from Euclidean division (i.e. long division) of A(x) by B(x).

A(x)	B(x)
R(x)	Q(x)

2° From the Euclidean relation $A(x) = B(x) \cdot Q(x) + R(x)$, we deduce the standard form of the rule.

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{Q(x)}$$

1. Perform the Euclidean division of $A(x) = 5x - 3$ by $B(x) = 2x + 4$ and determine the quotient Q(x) and the remainder R(x).

5x − 3	**2x + 4**
− 5x + 10	**5/2**
−13	

Q(x) = $\frac{5}{2}$***; R(x) = −13***

2. Deduce the standard form of the rule of the function $y = \frac{5x-3}{2x+4}$. ***y =*** $\frac{5}{2} + \frac{-13}{2(x+2)}$

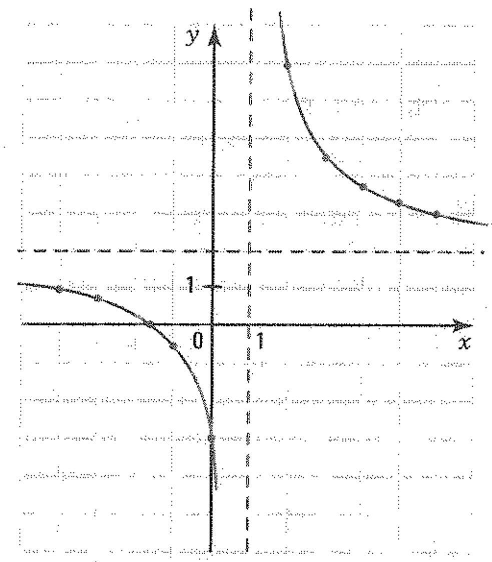
RATIONAL FUNCTION – GENERAL FORM

- The general form of a rational function is:

$$f(x) = \frac{ax+b}{cx+d}$$

- $\text{dom} f = \mathbb{R} \setminus \left\{-\frac{d}{c}\right\}$; $\text{ran} f = \mathbb{R} \setminus \left\{\frac{a}{c}\right\}$.
- Vertical asymptote: $x = -\frac{d}{c}$; horizontal asymptote: $y = \frac{a}{c}$.

Ex.: Given the rational function $f(x) = \frac{2x+3}{x-1}$.



– $\text{dom} f = \mathbb{R} \setminus \{1\}$; $\text{ran} f = \mathbb{R} \setminus \{2\}$.
– Vertical asymptote: $x = 1$;
horizontal asymptote: $y = 2$.
– Zero of f: $f(x) = 0 \Leftrightarrow 2x + 3 = 0 \Leftrightarrow x = -\frac{3}{2}$.
– Sign of f: $f(x) \geqslant 0 \Leftrightarrow x \in \left]-\infty, -\frac{3}{2}\right] \cup \left]1, +\infty\right[$
$f(x) \leqslant 0 \Leftrightarrow x \in \left[-\frac{3}{2}, 1\right[$.
– Variation of f: f is decreasing over $\mathbb{R} \setminus \{1\}$.
– f has no extrema.

13. Determine the domain and range of the following rational functions.

a) $y = \frac{3x+2}{x-5}$

dom $= \mathbb{R} \setminus \{5\}$, ***ran f*** $= \mathbb{R} \setminus \{3\}$

b) $y = \frac{-2x+4}{3x-6}$

dom $= \mathbb{R} \setminus \{2\}$, ***ran f*** $= \mathbb{R} \setminus \left\{-\frac{2}{3}\right\}$

c) $y = \frac{5x+4}{2x-3}$

dom $= \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$, ***ran f*** $= \mathbb{R} \setminus \left\{\frac{5}{2}\right\}$

14. Determine the zero (if it exists) and the initial value (if it exists) of the following functions.

a) $y = \frac{3x-2}{x-4}$

Zero: $\frac{2}{3}$, ***i.v:*** [illegible]

b) $y = \frac{-5x+10}{2x-5}$

Zero: 2, i.v: −2

c) $y = \frac{-2x-6}{4x}$

Zero: −3, i.v: does not exist

15. Determine over which interval the following functions are positive.

a) $y = \frac{4x+2}{x-3}$

f(x) $\geqslant 0$ ***over*** $\left]-\infty, \frac{-1}{2}\right] \cup \left]3, +\infty\right[$

b) $y = \frac{-2x+8}{4x-2}$

f(x) $\geqslant 0$ ***over*** $\left]\frac{1}{2}, 4\right]$

16. Study the variation of the following functions.

a) $y = \frac{-4x+9}{x-3}$

f ↗ ***over*** $\mathbb{R} \setminus \{3\}$

b) $y = \frac{2x+5}{3x-2}$

f ↘ ***over*** $\mathbb{R} \setminus \left\{\frac{2}{3}\right\}$

17. Write the rule of the following rational functions in general form.

a) $y = \frac{3}{2(x-1)} + 4$ $y = \frac{8x-5}{2x-2}$

b) $y = \frac{-2}{5(x-3)} - 1$ $y = \frac{-5x+13}{5x-15}$

18. Write the rule of the following rational functions in standard form.

a) $y = \frac{3x+2}{x-3}$ $y = \frac{11}{x-3} + 3$

b) $y = \frac{4x+3}{2x-6}$ $y = \frac{15}{2(x-3)} + 2$

c) $y = \frac{-2x+5}{3x+4}$ $y = \frac{23}{9\left(x+\frac{4}{3}\right)} - \frac{2}{3}$

19. Consider the rational functions $f(x) = \frac{2x+3}{x-4}$ and $g(x) = \frac{3x+5}{x+3}$.

a) Determine the rule of the composite

1. $g \circ f(x) =$ $y = \frac{11x-11}{5x-9}$

2. $f \circ g(x) =$ $y = \frac{9x+19}{-x-7}$

b) What can you say about the composition of a rational function with a rational function?

The composition of a rational function with a rational function is also a rational function.

20. Consider the rational function $y = \frac{5x+4}{x-3}$ (general form).

Justify the steps which enable you to determine the rule of the inverse f^{-1}.

1. Isolate x in the equation $y = \frac{5x+4}{x-3}$.

$y(x-3) = 5x+4$ *Cross products are equal.*

$xy - 3y = 5x + 4$ *Distributive property of multiplication over subtraction.*

$xy - 5x = 3y + 4$ *Subtract 5x and add 3y to each side.*

$x(y-5) = 3y + 4$ *Factor out x on the left side.*

$x = \frac{3y+4}{y-5}$ *Isolate the variable x.*

2. Switch the letters x and y to obtain the rule of the inverse.

We get: $y = \frac{3x+4}{x-5}$.

21. Consider the rational function $f(x) = \frac{3x-2}{2x+5}$.

a) Determine the rule of the inverse f^{-1}. $f^{-1}(x) = \frac{-5x-2}{2x-3}$

b) Verify that

1. $f \circ f^{-1}(x) = x$

2. $f^{-1} \circ f(x) = x$

22. Consider the rational function $f(x) = \frac{-2x+3}{4x+1}$.

a) Determine the domain and range of f. $\text{dom } f = \mathbb{R} \setminus \left\{-\frac{1}{4}\right\}$, $\text{ran } f = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$

b) Determine the rule of the inverse f^{-1}. $f^{-1}(x) = \frac{-x+3}{4x+2}$

c) Determine the domain and range of the inverse f^{-1} and verify that dom f^{-1} = ran f and ran f^{-1} = dom f.

$\text{dom } f^{-1} = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$, $\text{ran } f^{-1} = \mathbb{R} \setminus \left\{-\frac{1}{4}\right\}$

Evaluation 3

1. Determine the domain and range of each of the following functions.

a) $y = -2x^2 + 4x - 9$

dom = ℝ, ran =]−∞, −7]

b) $y = 4|x - 5| + 8$

dom = ℝ, ran = [8, +∞[

c) $y = \frac{1}{2}\sqrt{-(x-4)} + 3$

dom =]−∞, 4], ran = [3, +∞[

d) $y = -2\left[\frac{1}{3}(x-5)\right] + 4$

dom = ℝ,
ran = {y | y = −2m + 4, m ∈ ℤ}

e) $y = \frac{4}{3(x-1)} + 2$

dom = ℝ \ {1}, ran = ℝ \ {2}

f) $y = -4x + 2$

dom = ℝ, ran = ℝ

2. Determine the zero(s) and the initial value of each of the following functions.

a) $y = -2(x-4)^2 + 8$

Zeros: 2 and 6, i.v.: −24

b) $y = 3x - 5$

Zero: $\frac{5}{3}$, i.v.: −5

c) $y = \frac{3}{4}\sqrt{x+1} - 3$

Zero: 15, i.v.: $\frac{-9}{4}$

d) $y = 3\left[\frac{1}{2}(x-5)\right] + 6$

Zeros: [1, 3[, i.v.: −3

e) $y = \frac{-2}{5(x-1)} + 4$

Zero: $\frac{11}{10}$, i.v.: $\frac{22}{5}$

f) $y = 3|2x - 1| - 6$

Zeros: $\frac{-1}{2}$ and $\frac{3}{2}$, i.v.: −3

3. Determine over what interval each of the following functions is negative.

a) $y = 2x^2 - 5x - 3$

$\left]-\frac{1}{2}, 3\right[$

b) $y = -7x + 63$

]9, +∞[

c) $y = 2|8 - x| - 12$

]2, 14[

d) $y = \frac{2}{x-5} + 4$

$\left]\frac{9}{2}, 5\right[$

e) $y = -2\sqrt{6-x} + 4$

]−∞, 2[

f) $y = -\left[\frac{x}{2}\right] - 3$

[−6, +∞[

4. Determine over what interval each of the following functions is increasing.

a) $y = -3(x-5)(x+1)$

]−∞, 2]

b) $y = 2x - 5$

ℝ

c) $y = -[6 - 3x] + 1$

ℝ

d) $y = -3\sqrt{-(x-1)} + 4$

]−∞, 1]

e) $y = 3|x - 5| + 2$

[5, +∞[

f) $y = \frac{3}{2(x-1)} + 5$

∅

5. Determine, if it exists, the extremum of each of the following functions.

a) $y = -3x^2 + 12x - 7$

max = 5

b) $y = -2|3 - 2x| + 5$

max = 5

c) $y = -2\sqrt{x} + 7$

max = 7

6. Find the rule of the inverse of each of the following functions.

a) $y = -3x + 8$

$y = -\frac{1}{3}x + \frac{8}{3}$

b) $y = 3\sqrt{2-x} + 4$

$y = -\frac{1}{9}(x-4)^2 + 2,\ x \geq 4$

c) $y = \frac{3}{2(x-1)} + 8$

$y = \frac{3}{2(x-8)} + 1$

7. Consider the following real functions.

$f(x) = 3x - 8$ $g(x) = 3\sqrt{2x+1} - 5$ $h(x) = -2|x - 4| + 12$

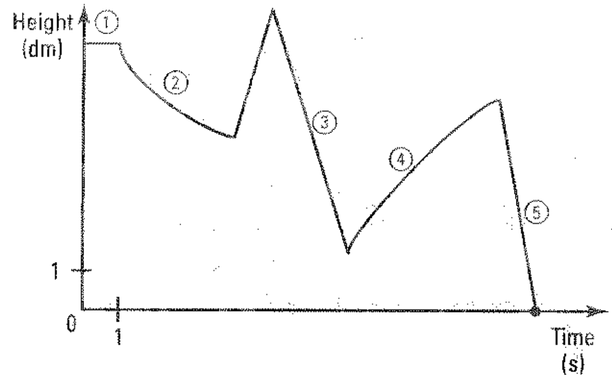
$i(x) = 3(x - 2)^2 + 4$ $k(x) = \frac{2}{x-5} + 1$ $l(x) = 3\left[\frac{1}{5}(x + 4)\right] - 6$

Determine

a) $f \circ g(4) =$ *4* b) $l \circ h(3) =$ *0* c) $k \circ i(5) =$ $\frac{14}{13}$

d) $f \circ l(0) =$ *-26* e) $k \circ f \circ h(2) =$ $\frac{13}{11}$ f) $l \circ h(-6) =$ *-9*

8. The path of a marble in a child's game can be represented by the graph in the Cartesian plane below. Initially, the marble is at a height of 7 dm from the ground.



$$f(x) = \begin{cases} 7 & \text{if } 0 \leq x < 1 \\ \frac{3}{x} + 4 & \text{if } 1 \leq x \leq 4 \\ a|x - 5| + 8 & \text{if } 4 \leq x \leq 7 \\ 2\sqrt{x - 7} + k & \text{if } 7 \leq x \leq 11 \\ -5.5x + b & \text{if } 11 \leq x \leq t \end{cases}$$

Determine the duration t of the marble's path.

$f(4) = 4.75$; $a|4 - 5| + 8 = 4.75$; $a = -3.25$

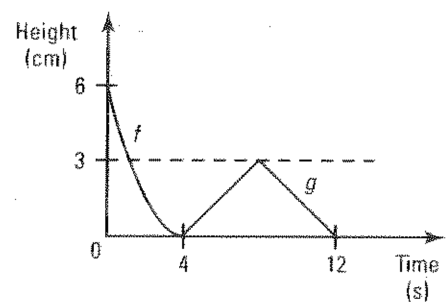
$f(7) = 1.5$; $k = 1.5$; $f(11) = 2\sqrt{11 - 7} + 1.5 = 5.5$

$-5.5(11) + b = 5.5$; $b = 66$; $-5.5t + 66 = 0 \Rightarrow t = 12$ s.

9. Aaron is playing an electronic game. The height of a flashing dot on the screen can be modeled by a square root function f from 0 to 4 seconds and by an absolute value function g from 4 to 12 seconds as indicated by the graph on the right.

The starting point of the flashing dot is the vertex of the function f.

Determine at what times the flashing dot is at a height of 1.5 cm.



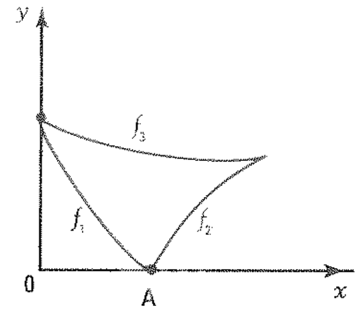
$f(x) = -3\sqrt{x} + 6$; $g(x) = -\frac{3}{4}|x - 8| + 3$

The flashing dot is at a height of 1.5 cm at the times $t = 2.25$ s, $t = 6$ s and $t = 10$ s.

10. A company's logo was drawn using the graphs of three square root functions as indicated in the figure on the right.

The rules of the functions f_1 and f_3 are respectively $f_1(x) = -\frac{4}{3}\sqrt{x} + 4$ and $f_3(x) = -\frac{1}{4}\sqrt{x} + 4$.

The x-coordinate of the intersection point of the functions f_2 and f_3 is 16. Knowing that point A is the vertex of the function f_2, what is the rule of the function f_2?



$f_3(16) = 3$; $A(9, 0)$; f_2: $y = a\sqrt{x - 9}$

The rule of the function f_2 is: $y = \frac{3}{7}\sqrt{7(x - 9)}$

11. The value of one Kandev share fluctuated, over a one-month period, according to the rule of an absolute value function. At the opening of the market, this share was worth \$3.50. Twelve days later, it reaches its maximum value of \$8.

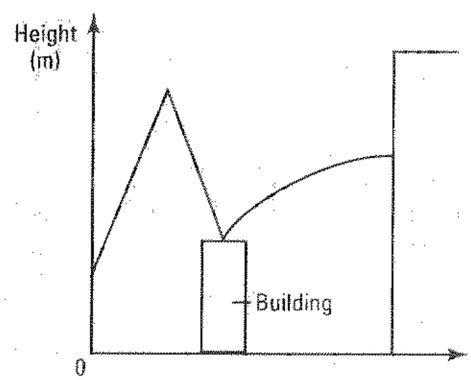
How many days go by between the moment the value of the share is worth \$5 for the first time and the moment it is worth \$2 on its descent?

$y = -\frac{3}{8}|x - 12| + 8$; $-\frac{3}{8}|x - 12| + 8 = 5$; $-\frac{3}{8}|x - 12| + 8 = 2$.

24 days.

12. The graph on the right illustrates a projectile's trajectory thrown from a height of 7 m. After 15 seconds, it reaches its maximum height of 40 m before descending onto the roof of an 18 m high building. The projectile bounces and, 4 seconds later, is at a height of 20 m. The first trajectory follows the model of an absolute value function and the second one follows the model of a square root function whose vertex corresponds to the point where it hits the roof of the building.

The projectile hits the wall of another building at a height of 25 m. How many seconds after the projectile was thrown does it hit the wall of the second building?



$y = -2{,}2\,|x - 15| + 40$; $-2{,}2\,|x - 15| + 40 = 18$; $y = \sqrt{x - 25} + 18$.

The projectile hits the wall of the second building 74 s after it is thrown.