Chapter 5

Trigonometry

CHALLENGE 5

1. If $P(t) = (a, b)$ is a trigonometric point, what are the coordinates of the trigonometric point $P(2t)$?

$P(2t) = (a^2 - b^2, 2ab)$

2. If $P(t) = (a, b)$ is a trigonometric point, what are the coordinates of the following trigonometric points?

a) $P(t + \pi)$ $(-a, -b)$ **b)** $P\left(t + \frac{\pi}{2}\right)$ $(-b, a)$

3. If $P(t) = (a, b)$ is a trigonometric point, determine

a) $\tan t$ $\frac{b}{a}$ **b)** $\text{cotan}\, t$ $\frac{a}{b}$ **c)** $\sec t$ $\frac{1}{a}$ **d)** $\csc t$ $\frac{1}{b}$

4. Determine the exact coordinates of the following trigonometric points.

a) $P\left(\frac{25\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **b)** $P\left(-\frac{19\pi}{4}\right)$ $\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ **c)** $P\left(-\frac{16\pi}{3}\right)$ $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

5. If $\tan t = \frac{12}{5}$ and $\pi \leq t \leq \frac{3\pi}{2}$, determine

a) $\sin t$ $\frac{-12}{13}$ **b)** $\cos t$ $\frac{-5}{13}$ **c)** $\sec t$ $\frac{-13}{5}$ **d)** $\csc t$ $\frac{-13}{12}$

6. Solve the equation $2 \sin \pi(x + 1) + 1 = 0$ in $\mathbb{R}$.

$S = \left\{\frac{-7}{6} + 2n\right\} \cup \left\{\frac{1}{6} + 2n\right\}$

7. Given the function $f(x) = -2 \cos \frac{\pi}{3}(x + 1) + 1$.

a) Determine

1. the amplitude of f. 2
2. the period of f. 6
3. the domain of f. $\mathbb{R}$
4. the range of f. $[-1, 3]$

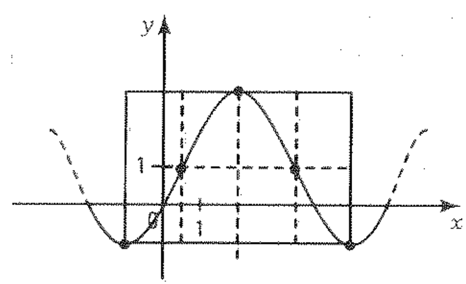
b) Graph the function f in the Cartesian plane.

c) Determine in $\mathbb{R}$

1. the zeros of f. $\{0 + 6n\} \cup \{4 + 6n\}$
2. the sign of f. $f(x) \geq 0$ over $[6n, 4 + 6n]$; $f(x) \leq 0$ over $[-1 + 6n, 6n] \cup [4 + 6n, 5 + 6n]$
3. the variation of f. $f \nearrow$ over $[-1 + 6n, 2 + 6n]$; $f \searrow$ over $[2 + 6n, 5 + 6n]$

d) Determine over $[11, 17]$

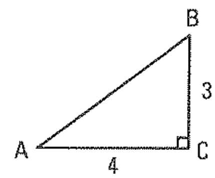
1. the zeros of f. 12 and 16
2. the sign of f. $f(x) \geq 0$ over $[12, 16]$; $f(x) \leq 0$ over $[11, 12] \cup [16, 17]$
3. the variation of f. $f \nearrow$ over $[11, 14]$; $f \searrow$ over $[14, 17]$



5.1 Trigonometric ratios in a right triangle

ACTIVITY 1 Trigonometric ratios

Consider the triangle ABC on the right.



a) 1. What is the length of the hypotenuse? *5 units*

2. What can we say about the acute angles A and B? *They are complementary.*

b) Determine the ratios

1. sin A $\frac{3}{5}$ 2. cos A $\frac{4}{5}$ 3. tan A $\frac{3}{4}$

c) By inverting each of the preceding ratios, we define the ratios

$\sec A = \frac{1}{\cos A}$, $\csc A = \frac{1}{\sin A}$ and $\cot A = \frac{1}{\tan A}$. Determine

1. sec A $\frac{5}{4}$ 2. csc A $\frac{5}{3}$ 3. cot A $\frac{4}{3}$

d) Verify the following trigonometric identities.

1. $\sin^2 A + \cos^2 A = 1$ $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$

2. $1 + \tan^2 A = \sec^2 A$ $1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16} = \sec^2 A$

3. $1 + \cot^2 A = \csc^2 A$ $1 + \left(\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9} = \csc^2 A$

TRIGONOMETRIC RATIOS

- All right triangles verify the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

- The acute angles of a right triangle are complementary.

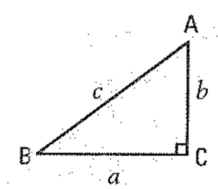
$$m\angle A + m\angle B = 90°$$

- We define the following ratios for the angle A.

$$\sin A = \frac{\text{measure of the opposite side}}{\text{measure of the hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{measure of the adjacent side}}{\text{measure of the hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{measure of the opposite side}}{\text{measure of the adjacent side}} = \frac{a}{b}$$

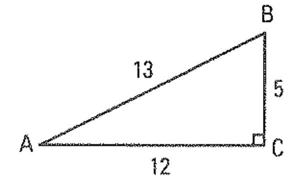


$\overline{AB}$ is the hypotenuse.
$\overline{BC}$ is the opposite side to ∠A.
$\overline{AC}$ is the adjacent side to ∠A.

By inverting the preceding ratios, we get:

$$\sec A = \frac{1}{\cos A} = \frac{c}{b} \qquad \csc A = \frac{1}{\sin A} = \frac{c}{a} \qquad \cot A = \frac{1}{\tan A} = \frac{b}{a}$$

1. Consider the right triangle ABC.



a) Determine the following ratios.

1. sin A $\frac{5}{13}$ 2. cos A $\frac{12}{13}$ 3. tan A $\frac{5}{12}$

4. sec A $\frac{13}{12}$ 5. csc A $\frac{13}{5}$ 6. cot A $\frac{12}{5}$

b) Verify the trigonometric identities.

1. $\sin^2 A + \cos^2 A = 1$ $\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

2. $1 + \tan^2 A = \sec^2 A$ $1 + \left(\frac{5}{12}\right)^2 = \left(\frac{13}{12}\right)^2$

3. $1 + \cot^2 A = \csc^2 A$ $1 + \left(\frac{12}{5}\right)^2 = \left(\frac{13}{5}\right)^2$

c) Verify that

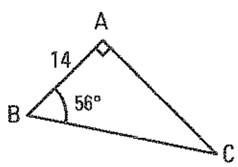
1. $\tan A = \frac{\sin A}{\cos A}$ $\frac{5}{12} = \frac{\frac{5}{13}}{\frac{12}{13}}$ 2. $\cot A = \frac{\cos A}{\sin A}$ $\frac{12}{5} = \frac{\frac{12}{13}}{\frac{5}{13}}$

d) Verify that

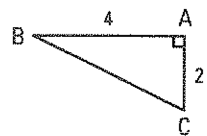
1. sin A = cos B 2. cos A = sin B 3. tan A = cot B

2. Solve the following triangles (round the measures of the sides and angles to the nearest tenth).

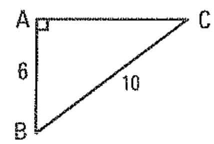
a)



b)



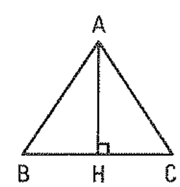
c)



a)	b)	c)
$m\overline{AC} = 20.8$	$m\overline{BC} = 4.5$	$m\overline{AC} = 8$
$m\overline{BC} = 25.0$	$m\angle B = 26.6°$	$m\angle B = 53.1°$
$m\angle C = 34°$	$m\angle C = 63.4°$	$m\angle C = 36.9°$

ACTIVITY 2 Remarkable angles: 0°, 30°, 45°, 60°, 90°

a) The triangle ABC on the right is equilateral, with each side measuring 1 unit. We have drawn the altitude AH.



1. Explain why $m\overline{BH} = 0.5$ u.
   In an equilateral triangle, the altitude AH is also a median.
2. Explain why m ∠ABC = m ∠BAC = m ∠ACB = 60°.
   In an equilateral triangle, each angle measures 60°.
3. Explain why m ∠BAH = 30°.
   In an equilateral triangle, the altitude AH is also a perpendicular bisector.
4. Refer to the triangle ABH to show that $\sin 30° = \frac{1}{2}$.
   $\sin 30° = \frac{0.5}{1} = \frac{1}{2}$

5. Explain why $m\overline{AH} = \frac{\sqrt{3}}{2}$.

$m\overline{AH}^2 + m\overline{BH}^2 = m\overline{AB}^2$ (Pythagoras) $\Rightarrow m\overline{AH}^2 = 1 - (0.5)^2 = \frac{3}{4} \Rightarrow m\overline{AH} = \frac{\sqrt{3}}{2}$

6. Refer to the triangle ABH to show that $\sin 60° = \frac{\sqrt{3}}{2}$.

$\sin 60° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

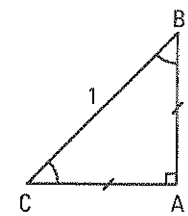
7. Explain why $\cos 60° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

$\cos 60° = \sin 30° = \frac{1}{2}$ (complementary angles); $\cos 30° = \sin 60° = \frac{\sqrt{3}}{2}$ (complementary angles)

8. Explain why $\tan 30° = \frac{\sqrt{3}}{3}$ and $\tan 60° = \sqrt{3}$.

$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$; $\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

b) The given right triangle is isosceles. The hypotenuse measures 1 unit.



1. What is the measure of the sides of the right angle?

Let $m\overline{AB} = m\overline{AC} = x$

We have: $x^2 + x^2 = 1$ (Pythagoras); $2x^2 = 1$; $x = \frac{\sqrt{2}}{2}$.

2. What is the measure of each acute angle? *45°*

3. Show that $\sin 45° = \frac{\sqrt{2}}{2}$ and that $\cos 45° = \frac{\sqrt{2}}{2}$.

$\sin 45° = \sin B = \frac{\sqrt{2}}{2}$; $\cos 45° = \cos B = \frac{\sqrt{2}}{2}$

4. Explain why $\tan 45° = 1$. *$\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$*

c) Using a calculator, verify that $\sin 0° = 0$; $\sin 90° = 1$; $\cos 0° = 1$ and $\cos 90° = 0$.

d) Explain why $\tan 90°$ is not defined.

$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0}$; Division by 0 is not defined.

REMARKABLE ANGLES

Angles	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

Memorize the sine line.

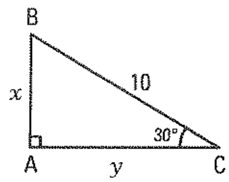
From the sine line, we deduce the cosine line, since $\cos x = \sin(90° - x)$.

From the sine and cosine lines, we deduce the tangent line, since $\tan x = \frac{\sin x}{\cos x}$.

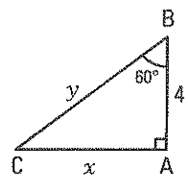
3. Complete the following table by giving the exact ratios.

Angles	0°	30°	45°	60°	90°
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	
Cosecant		2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent		$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

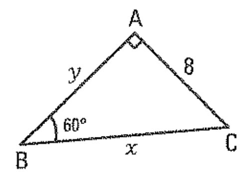
4. Determine the exact measures of x and y.

a) 

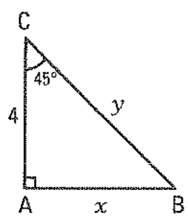
$x = 5;\ y = 5\sqrt{3}$

b) 

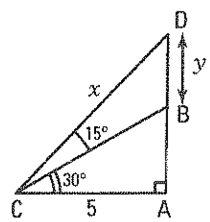
$x = 4\sqrt{3}\ ;\ y = 8$

c) 

$x = \frac{16\sqrt{3}}{3};\ y = \frac{8\sqrt{3}}{3}$

d) 

$x = 4;\ y = 4\sqrt{2}$

e) 

$x = 5\sqrt{2};\ y = 5 - \frac{5\sqrt{3}}{3}$

5.2 Arc length

ACTIVITY 1 Units of angle measures

A **turn** is a natural unit for measuring angles. Certain fractions of turns define units of angle measures.

– a **degree** (°) corresponds to the fraction $\frac{1}{360}$ of a turn. Therefore: 1 turn = 360°.

– a **gradian** (grad) corresponds to the fraction $\frac{1}{400}$ of a turn. Therefore: 1 turn = 400 grad.

– a **radian** (rad) corresponds to the fraction $\frac{1}{2\pi}$ of a turn (0.159... turn). Therefore: 1 turn = 2π rad.

a) Convert $\frac{1}{4}$ turn in

1. degrees *90°* 2. gradians *100 grad* 3. radians *$\frac{\pi}{2}$ rad*

b) Convert 180° in

1. turns *$\frac{1}{2}$ turn* 2. gradians *200 grad* 3. radians *π rad*

RADIANS

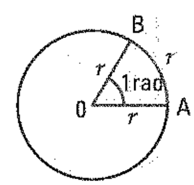
- In a circle, a radian (rad) corresponds to the measure of the central angle that subtends an arc whose length is equal to the circle's radius.

  We have: 1 turn = 360° = 2π rad.

  Remember that:

  $$\pi \text{ rad} = 180°$$

  Therefore: $1 \text{ rad} = \frac{180°}{\pi}$ and $1° = \frac{\pi}{180°}$ rad.



$m\overline{0A} = m\overline{0B} = m\widehat{AB} = r$
$m\ \angle A0B = 1$ rad

  These relations enable you to convert radians into degrees and vice-versa.

  Ex.: $3 \text{ rad} = 3\left(\frac{180°}{\pi}\right) = \frac{540°}{\pi}$.

  $120° = 120\left(\frac{\pi}{180°}\text{ rad}\right) = \frac{2\pi}{3}$ rad.

- When the unit measure of an angle is not indicated, it is implied that the angle measure is in radians.

1. Express, in radians, the measures of the following angles.

a) 30° *$\frac{\pi}{6}$ rad* **b)** 45° *$\frac{\pi}{4}$ rad* **c)** 60° *$\frac{\pi}{3}$ rad* **d)** 90° *$\frac{\pi}{2}$ rad*

e) 120° *$\frac{2\pi}{3}$ rad* **f)** 135° *$\frac{3\pi}{4}$ rad* **g)** 300° *$\frac{5\pi}{3}$ rad* **h)** 390° *$\frac{13\pi}{6}$ rad*

2. Express, in degrees, the measures of the following angles.

a) 2 rad *$\frac{360°}{\pi}$* **b)** $\frac{\pi}{3}$ rad *60°* **c)** $\frac{7\pi}{6}$ rad *210°* **d)** $\frac{4\pi}{3}$ rad *240°*

e) 5 rad *$\frac{900°}{\pi}$* **f)** $\frac{2\pi}{5}$ rad *72°* **g)** $\frac{11\pi}{6}$ rad *330°* **h)** $\frac{5\pi}{2}$ rad *450°*

3. Convert according to the desired unit.

a) 0.5 turn = $\underline{\pi}$ rad b) 60° = $\underline{\frac{1}{6}}$ turn c) $\frac{5\pi}{6}$ rad = $\underline{150°}$

d) $\frac{3\pi}{2}$ rad = $\underline{\frac{3}{4}}$ turn e) 150° = $\underline{\frac{5\pi}{6}}$ rad f) $\frac{2\pi}{3}$ rad = $\underline{\frac{1}{3}}$ turn

g) 2 turns = $\underline{720°}$ h) 240° = $\underline{\frac{4\pi}{3}}$ rad i) 405° = $\underline{\frac{9\pi}{4}}$ rad

4. Express, in radians and in terms of π, the measures of the angles that are multiples of 45°.

a) 45° $\frac{\pi}{4}$ ***rad*** b) 90° $\frac{\pi}{2}$ ***rad*** c) 135° $\frac{3\pi}{4}$ ***rad*** d) 180° π ***rad***

e) 225° $\frac{5\pi}{4}$ ***rad*** f) 270° $\frac{3\pi}{2}$ ***rad*** g) 315° $\frac{7\pi}{4}$ ***rad*** h) 360° 2π ***rad***

5. The following angles, in radians, are expressed in terms of $\frac{\pi}{6}$. Express them in degrees.

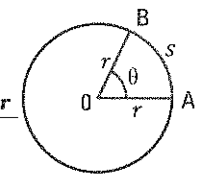
a) $\frac{\pi}{6}$ rad ***30°*** b) $\frac{5\pi}{6}$ rad ***150°*** c) $\frac{7\pi}{6}$ rad ***210°*** d) $\frac{11\pi}{6}$ rad ***330°***

6. The following angles, in radians, are expressed in terms of $\frac{\pi}{3}$. Express them in degrees.

a) $\frac{\pi}{3}$ rad ***60°*** b) $\frac{2\pi}{3}$ rad ***120°*** c) $\frac{4\pi}{3}$ rad ***240°*** d) $\frac{5\pi}{3}$ rad ***300°***

ACTIVITY 2 Arc length

Consider the circle on the right with radius r and central angle A0B.



a) What is the measure of the arc AB if the central angle A0B measures

1. 1 radian? ***r*** 2. 2 radians? ***2r*** 3. 3 radians? ***3r***

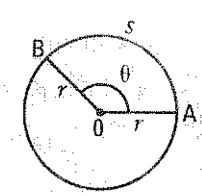
b) If we let θ represent the measure, in radians, of the central angle A0B and s represent the measure of the arc AB, express s as a function of r and θ. ***s = r θ***

c) If the circle has a radius of $r = 1$, what can be said of the measure s of the arc subtended by the central angle measuring θ radians?

s = θ, the length of the arc is equal to the measure of the central angle which subtends this arc.

ARC LENGTH

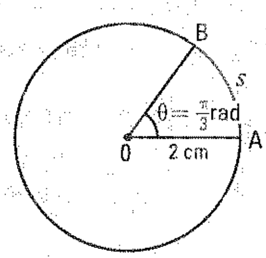
- Consider a circle with radius r and a central angle A0B which subtends the arc AB.



  If θ represents the measure, in radians, of the central angle and s represents the length of the subtended arc, then

$$s = r\,\theta$$

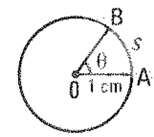
- Note that r and s are expressed in the same units.



  Ex.: On the right, we have: $r = 2$ cm and $\theta = \frac{\pi}{3}$ rad.

  The arc AB measures $s = \frac{2\pi}{3}$ cm ≈ 2.09 cm.

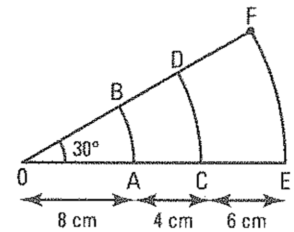
- In a circle with a radius of 1 unit, the length of an arc is equal to the central angle which subtends this arc.



$$r = 1 \Rightarrow s = \theta$$

Ex.: $r = 1$ cm and $\theta = \frac{\pi}{3}$ rad $\Rightarrow$ $s = \frac{\pi}{3}$ cm ≈ 1.05 cm.

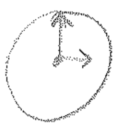
7. By referring to the figure on the right, calculate the length of



a) $\overset{\frown}{AB}$ ***$\frac{4}{3}\pi$ cm*** b) $\overset{\frown}{CD}$ ***2π cm*** c) $\overset{\frown}{EF}$ ***3π cm***

8. In a circle with radius r, a central angle A0B measuring θ subtends the arc AB of length s.

Complete the table of values on the right.



r = 0,8 cm
C = 2πr = 1,6π cm ≈ 5,03 cm
2,25 × 1,6π = 3,6π cm = 11,31 cm

r	θ	s
2 cm	$\frac{2\pi}{3}$ rad	***$\frac{4\pi}{3}$ cm***
6 cm	$\frac{5\pi}{6}$ rad	5π cm
12 cm	***$\frac{5\pi}{6}$ rad***	10π cm

9. The radius of curvature of a railroad track is equal to 600 m. What is the central angle of a train's trajectory if it travels 1.8 km along this track? ***θ = 3 rad***

1800m

10. The minute hand of a watch measures 0.8 cm. What is the distance traveled by the end of the minute hand in 11.31 cm 24 × 1.6π = 38.4π 490.3

a) 2 h 15 min? ***0.47 cm*** b) 1 day? ***120.6 cm*** ✓ c) 1 year? ***440.2 m***

2.25

11. The wheel of a bicycle turns at a speed of 150 turns per minute. 150(2π) = 300π rpm = 5π rad/s

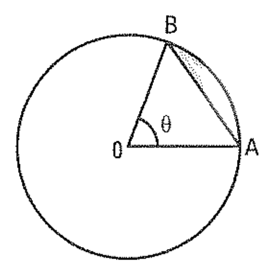
a) Express the speed in radians per second. ***5π rad/s***

b) What is the distance traveled in 30 minutes if the radius of the wheel measures 40 cm?

3600π m ≈ 11 310 m ✓ C = 80π cm 150 × 30 = 4500 rev. × 80π

12. A circular segment is the surface on a disk between an arc and the chord which subtends this arc. Let r represent the radius of the circle on the right and θ represent the central angle that defines the circular segment.

a) Show that the area A of the circular segment is: $A = \frac{r^2}{2}(\theta - \sin\theta)$.



A = area of the circular sector A0B – area of △A0B.

Area of the circular sector A0B = $\frac{\theta}{2\pi}(\pi r^2) = \frac{\theta r^2}{2}$; area △A0B = $\frac{r^2}{2}\sin\theta$.

$A = \frac{r^2}{2}(\theta - \sin\theta)$.

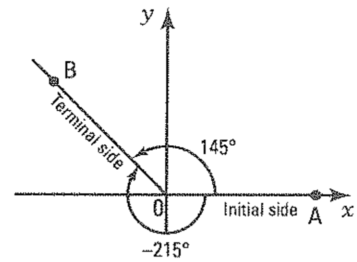
b) Calculate the area A of a circular segment when $r = 10$ cm and $\theta = \frac{\pi}{6}$.

$A = \frac{10^2}{2}\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right) = 50\left(\frac{\pi}{6} - \frac{1}{2}\right) = \frac{25\pi}{3} - 25 \approx 1.18$ cm².

5.3 Trigonometric circle

Activity 1 Trigonometric angle

In the Cartesian plane on the right, the angle A0B is obtained by having the **initial** side 0A undergo a rotation with centre 0, angle of 145° in a positive direction (counter-clockwise). Such an angle is called a **trigonometric angle**. The side 0B is called the **terminal** side.



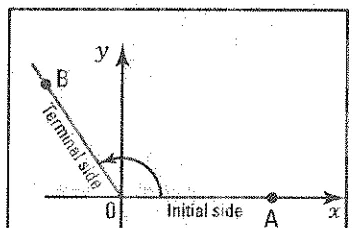
Define a clockwise rotation which applies the initial side 0A onto the terminal side 0B.

A rotation, centered at 0, in the clockwise direction and of angle 215°.

TRIGONOMETRIC ANGLE

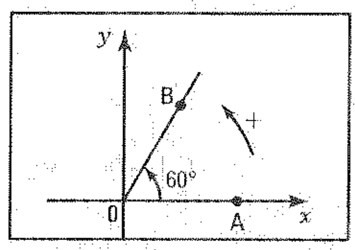
In a **trigonometric angle**, we distinguish:

- the **vertex**, located at the centre 0 of the Cartesian plane.
- the side of the angle, called **initial side**, that is along the positive x-axis.
- the other side of the angle, called **terminal side**, obtained by a rotation centered at 0 of the initial side.

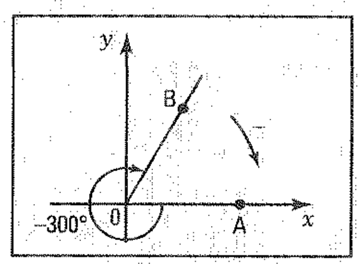


∠A0B is trigonometric

- If the rotation is performed in a **positive** direction (counter-clockwise), then the measure of the trigonometric angle is **positive**.



- If the rotation is performed in a **negative** direction (clockwise), then the measure of the trigonometric angle is **negative**.



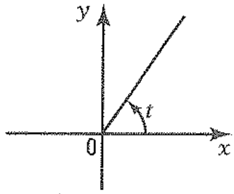
1. Consider the trigonometric angle A0B. What conditions must be met when considering

a) the vertex 0 of the angle? *The vertex 0 is the origin of the Cartesian plane.*

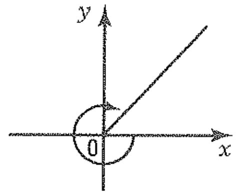
b) the point A if $\overline{0A}$ is the initial side of the angle?
The point A must be located on the positive x-axis.

c) the point B if $\overline{0B}$ is the terminal side of the angle? *None*

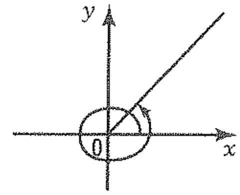
2. Determine the measure t, in degrees and radians, of each of the following trigonometric angles.

a)


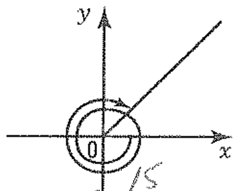
45°; $\frac{\pi}{4}$ rad

b)


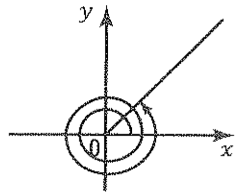
–315°; $\frac{-7\pi}{4}$ rad

c)


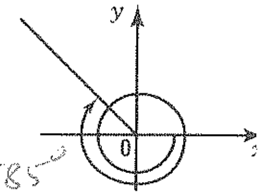
405°; $\frac{9\pi}{4}$ rad

d)


–675° 15 *~~–405°~~; $-\frac{9\pi}{4}$ rad*

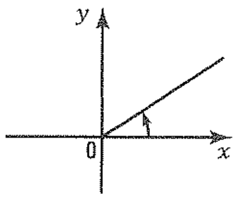
e)


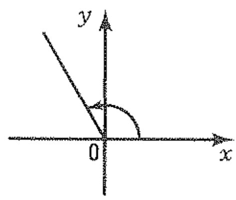
765°; $\frac{17\pi}{4}$ rad

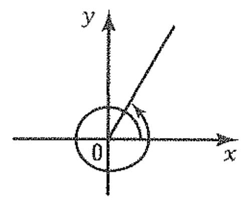
f)


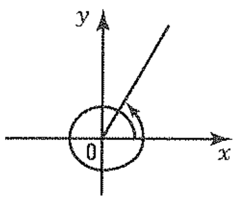
–585° *~~–565°~~; $\frac{-13\pi}{4}$ rad*

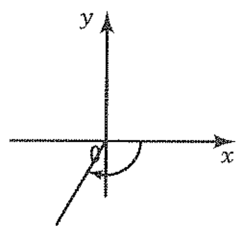
3. Represent each of the following trigonometric angles with the given measure.

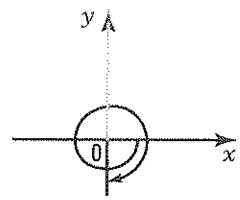
a) $\frac{\pi}{6}$ rad


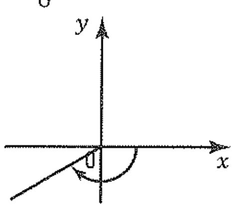
b) 120°


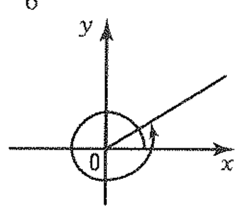
c) 420°


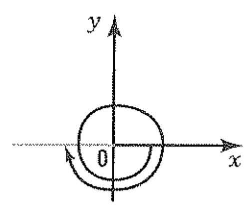
d) $\frac{7\pi}{3}$ rad


e) –120°


f) –450°


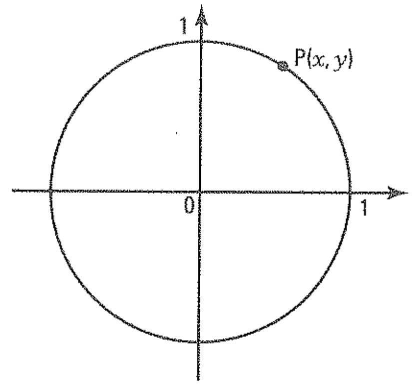
g) $-\frac{5\pi}{6}$ rad


h) $\frac{13\pi}{6}$ rad


i) -3π rad


ACTIVITY 2 Trigonometric circle and trigonometric points

A circle centered at 0 with radius 1 has been drawn in the Cartesian plane on the right. This circle is called the **trigonometric circle**. Any point P(x, y) on this circle is called a **trigonometric point**.



Any trigonometric point P(x, y) verifies the equation $x^2 + y^2 = 1$ and, conversely, any point P(x, y) that verifies the equation $x^2 + y^2 = 1$ is trigonometric.

Determine if the following points are trigonometric.

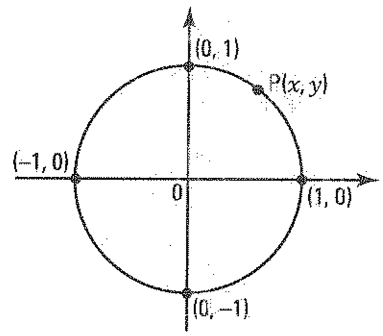
a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ ***No*** **b)** $(1, 0)$ ***Yes*** **c)** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ ***Yes***

d) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ ***Yes*** **e)** $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$ ***Yes*** **f)** $(0, -1)$ ***Yes***

TRIGONOMETRIC CIRCLE AND TRIGONOMETRIC POINTS

- The **trigonometric circle** is a circle centered at 0, the origin of the Cartesian plane, with a radius of 1.
- Any point P(x, y) on the unit circle is called a **trigonometric point**. We have:

$$P(x, y) \text{ is trigonometric} \Leftrightarrow x^2 + y^2 = 1$$



4. The point $\left(\frac{a-4}{13}, \frac{a+3}{13}\right)$ is a trigonometric point. Determine the possible values for a.

$\left(\frac{a-4}{13}\right)^2 + \left(\frac{a+3}{13}\right)^2 = 1$ $(a-4)^2 + (a+3)^2 = 169 \Leftrightarrow 2a^2 - 2a - 144 = 0 \Leftrightarrow a = 9$ or $a = -8$.

5. Determine the possible values for x if the following points are trigonometric.

a) P$(x, 1)$ $x = 0$ **b)** P$\left(x, \frac{1}{2}\right)$ $x = -\frac{\sqrt{3}}{2}$ or $x = \frac{\sqrt{3}}{2}$

c) P$(x, 0.6)$ $x = -0.8$ or $x = 0.8$ **d)** P$\left(x, \frac{5}{13}\right)$ $x = -\frac{12}{13}$ or $x = \frac{12}{13}$

6. In each of the following cases, determine the coordinates of the trigonometric point P.

a) P$\left(\frac{1}{2}, y\right) \in$ 4th quadrant. $P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

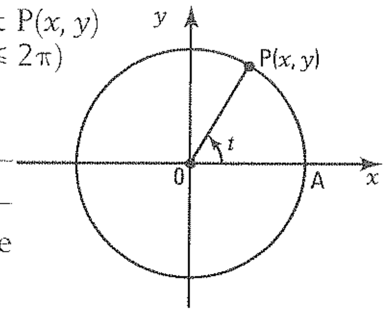
b) P$(-0.6, y) \in$ 2nd quadrant. $P(-0.6, 0.8)$

c) P$\left(\frac{\sqrt{3}}{2}, y\right) \in$ 1st quadrant. $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

d) P$\left(-\frac{1}{2}, y\right) \in$ 3rd quadrant. $P\left(-\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$

ACTIVITY 3 Locating a trigonometric point

Consider the trigonometric circle on the right, a trigonometric point P(x, y) and the measure t, in radians, of the trigonometric angle A0P. ($0 \leqslant t \leqslant 2\pi$)

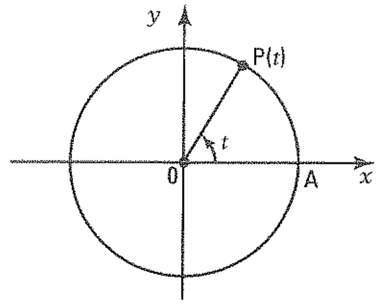


a) What is the length of arc AP? Justify your answer.

The unit circle has a radius of r = 1 unit. The length s of the arc AP is therefore s = rt = 1 × t = t units.

b) Let P(t) represent the trigonometric point associated with the trigonometric angle t. ($t \in \mathbb{R}$)

Determine the Cartesian coordinates (x, y) of the following trigonometric points P(t).



1. P(0) ***(1, 0)*** 2. $P\left(\frac{\pi}{2}\right)$ ***(0, 1)*** 3. P(π) ***(-1, 0)***

4. $P\left(\frac{3\pi}{2}\right)$ ***(0, -1)*** 5. P(2π) ***(1, 0)*** 6. $P\left(-\frac{\pi}{2}\right)$ ***(0, -1)***

7. P(−π) ***(-1, 0)*** 8. $P\left(-\frac{3\pi}{2}\right)$ ***(0, 1)*** 9. P(−2π) ***(1, 0)***

c) Is it true to say that for each real number t, there is a unique corresponding trigonometric point on the trigonometric circle? ***Yes***

d) Can we say that each trigonometric point P(x, y) on the trigonometric circle corresponds to a unique trigonometric angle t? ***No***

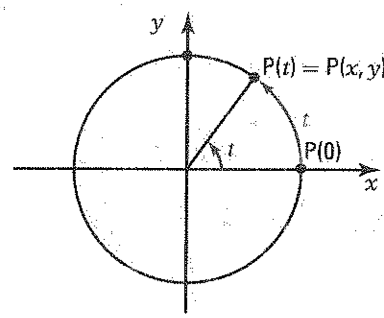
LOCATING A TRIGONOMETRIC POINT

- Each real number t corresponds to a unique point on the unit circle written as P(t).

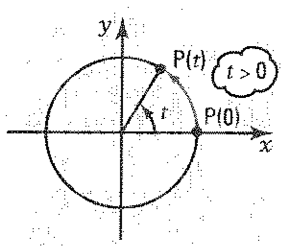
  P(t) is the extremity of the arc whose origin is the point P(0) and whose directed measure is equal to t.

  P(x, y) is the Cartesian notation of this point.

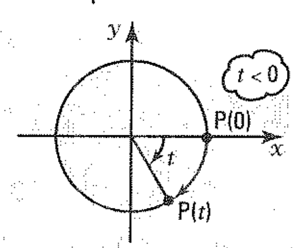
  Ex.: P(0°) = P(1, 0)
  P(90°) = P(0, 1)



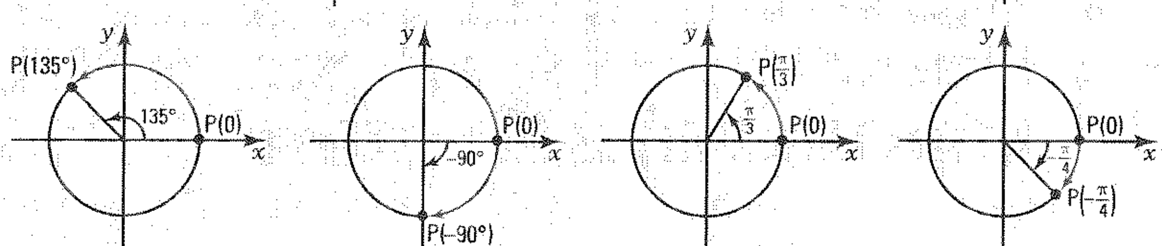
- If t is **positive**, we locate the point P(t) by moving in a **counter-clockwise** direction.



- If t is **negative**, we locate the point P(t) by moving in a **clockwise** direction.



Ex.:

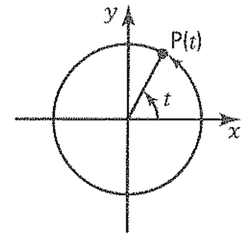


7. Indicate in which quadrant each of the following trigonometric points is located.

a) P(120°) *II* **b)** P(300°) *IV* **c)** P(400°) *I* **d)** P(−120°) *III*

e) $P\left(\frac{5\pi}{6}\right)$ *II* **f)** $P\left(\frac{4\pi}{3}\right)$ *III* **g)** $P\left(\frac{11\pi}{6}\right)$ *IV* **h)** $P\left(-\frac{3\pi}{4}\right)$ *III*

8. If $P(t)$ is a trigonometric point located in the 1st quadrant, deduce the quadrant that each of following trigonometric points will be located in.

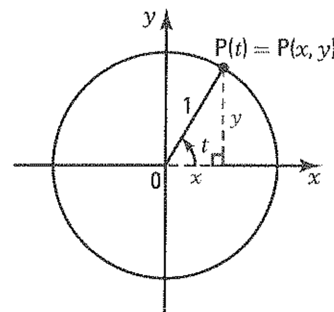


a) $P(t+\pi)$ *III* **b)** $P\left(t+\frac{\pi}{2}\right)$ *II* **c)** $P\left(t+\frac{3\pi}{2}\right)$ *IV*

d) $P(-t)$ *IV* **e)** $P(\pi - t)$ *II* **f)** $P\left(t-\frac{\pi}{2}\right)$ *IV*

g) $P(t+2\pi)$ *I* **h)** $P(t-2\pi)$ *I* **i)** $P(t+6\pi)$ *I*

ACTIVITY 4 Cartesian coordinates of a trigonometric point



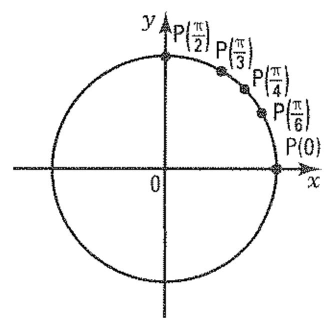
a) Consider a trigonometric point P(t) located in the 1st quadrant; $\left(0 \leqslant t \leqslant \frac{\pi}{2}\right)$.

Which trigonometric ratio enables you to calculate

1. the x-coordinate of the point $P(t)$? *x = cos t*
2. the y-coordinate of the point $P(t)$? *y = sin t*

b) Determine, without using a calculator, the coordinates of the trigonometric points

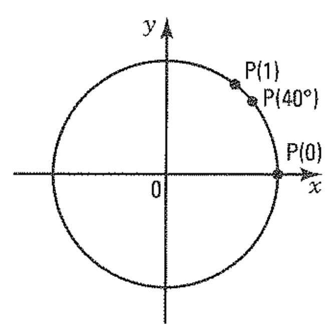
1. $P(0) =$ *(cos 0, sin 0) = (1, 0)*
2. $P\left(\frac{\pi}{2}\right) =$ $\left(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right) = (0, 1)$



c) The points $P\left(\frac{\pi}{6}\right)$, $P\left(\frac{\pi}{4}\right)$ and $P\left(\frac{\pi}{3}\right)$ are trigonometric points in the 1st quadrant called **remarkable trigonometric points**. Determine the exact coordinates of these points without using a calculator.

1. $P\left(\frac{\pi}{6}\right) = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
2. $P\left(\frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. $P\left(\frac{\pi}{3}\right) = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

d) The points P(1 rad) and P(40°) are trigonometric points located in the 1st quadrant. Using a calculator, determine the coordinates of these points to the nearest hundredth. Use the appropriate mode, rad or deg, which applies.



1. P(1) = *(cos 1, sin 1) = (0.54, 0.84)*
2. P(40°) = *(cos 40, sin 40) = (0.77, 0.64)*

e) The results observed in a), valid for any trigonometric point in the 1st quadrant, are generalized and bring us to the definition of the Cartesian coordinates of any trigonometric point P(t).

We have: P(t) = (cos t, sin t) where $0 \leqslant t \leqslant 2\pi$.

Calculate, using a calculator, the Cartesian coordinates of

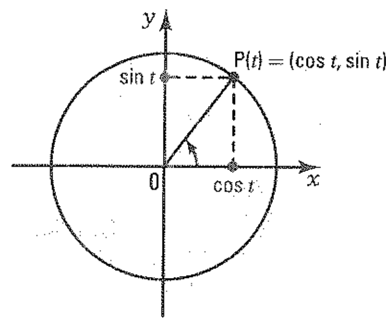
1. P(100°) ***(–0.17, 0.98)***   2. P(200°) ***(–0.94, –0.34)***   3. P(300°) ***(0.5, –0.87)***

CARTESIAN COORDINATES OF A TRIGONOMETRIC POINT

- Given a trigonometric point P(t). ($0 \leqslant t \leqslant 2\pi$)
  - the x-coordinate of P(t) is equal to **cos** t.
  - the y-coordinate of P(t) is equal to **sin** t.

Note that:

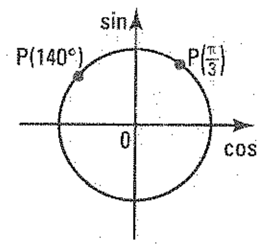
$$P(t) = (\cos t, \sin t)$$



By convention, we call the x-axis the **cosine axis**, and the y-axis the **sine axis**.

Ex.: $P\left(\frac{\pi}{3}\right) = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

P(140°) = (cos 140°, sin 140°) = (–0.7660, 0.6428)



9. For each of the following trigonometric points P(t), indicate
1. the quadrant in which the trigonometric point is located.
2. the sign of cos t and the sign of sin t.

a) P(160°)
1. ***II***
2. ***(–, +)***

b) P(350°)
1. ***IV***
2. ***(+, –)***

c) P(–150°)
1. ***III***
2. ***(–, –)***

d) P(750°)
1. ***I***
2. ***(+, +)***

e) $P\left(\frac{5\pi}{6}\right)$
1. ***II***
2. ***(–, +)***

f) $P\left(\frac{5\pi}{3}\right)$
1. ***IV***
2. ***(+, –)***

g) $P\left(-\frac{2\pi}{3}\right)$
1. ***III***
2. ***(–, –)***

h) $P\left(\frac{10\pi}{3}\right)$
1. ***III***
2. ***(–, –)***

10. Using a calculator, determine the coordinates of the following trigonometric points to the nearest thousandth.

a) P(175°) ***(–0.996, 0.087)***   **b)** P(625°) ***(–0.087, –0.996)***

c) $P\left(\frac{11\pi}{5}\right)$ ***(0.809, 0.588)***   **d)** $P\left(-\frac{29\pi}{6}\right)$ ***(–0.866, –0.5)***

11. Knowing that $P(t) = \left(\frac{3}{5}, \frac{-4}{5}\right)$ is a trigonometric point, determine

a) cos t = $\frac{3}{5}$ **b)** sin t = $-\frac{4}{5}$ **c)** tan t = $-\frac{4}{3}$

d) sec t = $\frac{5}{3}$ **e)** csc t = $-\frac{5}{4}$ **f)** cotan t = $-\frac{3}{4}$

12. Knowing that $P(t) = \left(\cos t, \frac{5}{13}\right)$ is a trigonometric point located in the 2nd quadrant, determine

a) $\cos t =$ $\frac{-12}{13}$ **b)** $\sec t =$ $\frac{-13}{12}$ **c)** $\csc t =$ ~~$\frac{5}{13}$~~ 13/5

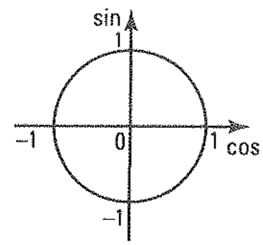
d) $\tan t =$ $\frac{-5}{12}$ **e)** $\cot t =$ $\frac{-12}{5}$

13. For each of the following trigonometric points, give the two possible values for the missing coordinate.

a) $P\left(\frac{1}{2}, \ldots\right)$ $\pm\frac{\sqrt{3}}{2}$ **b)** $P\left(\ldots, \frac{\sqrt{3}}{2}\right)$ $\pm\frac{1}{2}$ **c)** $P(\ldots, 0.6)$ ± 0.8

d) $P\left(\frac{-5}{13}, \ldots\right)$ $\pm\frac{12}{13}$ **e)** $P\left(\frac{2}{3}, \ldots\right)$ $\pm\frac{\sqrt{5}}{3}$ **f)** $P\left(\ldots, \frac{\sqrt{2}}{2}\right)$ $\pm\frac{\sqrt{2}}{2}$

14. A trigonometric point $P(t)$ has an x-coordinate of $\cos t = 0.8$.



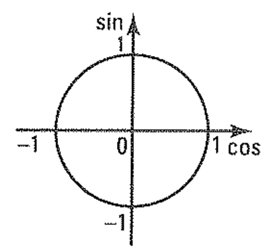
a) If the point $P(t)$ is located in the 1st quadrant,

1. determine the y-coordinate $\sin t$. $\sin t = 0.6$
2. deduce, in degrees, the value of t knowing that $0 \leqslant t \leqslant 90°$.
   $t = 36.9°$ ✓
3. deduce, in degrees, the value of t knowing that $360° \leqslant t \leqslant 450°$.
   $t = 396.9°$ ✓

b) If the point $P(t)$ is located in the 4th quadrant,

1. determine the y-coordinate $\sin t$. $\sin t = -0.6$
2. deduce, in degrees, the value of t knowing that $270° \leqslant t \leqslant 360°$. $t = 323.1°$
3. deduce, in degrees, the value of t knowing that $630° \leqslant t \leqslant 720°$. $t = 683.1°$

15. A trigonometric point $P(t)$ has an x-coordinate of $\cos t = -0.6$.



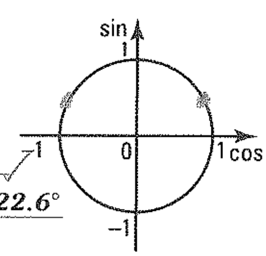
a) If the point $P(t)$ is located in the 2nd quadrant,

1. determine the y-coordinate $\sin t$. $\sin t = 0.8$
2. deduce, in degrees, the value of t knowing that $90° \leqslant t \leqslant 180°$.
   $t =$ ~~$53.1°$~~ 126.9°
3. deduce, in degrees, the value of t knowing that $450° \leqslant t \leqslant 540°$.
   $t =$ ~~$413.1°$~~ 486.9°

b) If the point $P(t)$ is located in the 3rd quadrant,

1. determine the y-coordinate $\sin t$. $\sin t = -0.8$
2. deduce, in degrees, the value of t knowing that $180° \leqslant t \leqslant 270°$. $t = 233.1°$
3. deduce, in degrees, the value of t knowing that $540° \leqslant t \leqslant 630°$. $t = 593.1°$

16. A trigonometric point $P(t)$ has a y-coordinate of $\sin t = \frac{5}{13}$.



a) If the point $P(t)$ is located in the 1st quadrant,

1. determine the x-coordinate $\cos t$. $\cos t = \frac{12}{13}$ ✓
2. deduce, in degrees, the value of t knowing that $0° \leqslant t \leqslant 90°$. $t = 22.6°$

b) If the point $P(t)$ is located in the 2nd quadrant,

1. determine the x-coordinate $\cos t$. $\cos t = \frac{-12}{13}$
2. deduce, in degrees, the value of t knowing that $90° \leqslant t \leqslant 180°$. $t = 157.4°$

17. A trigonometric point P(t) has a y-coordinate of $\sin t = \frac{-4}{5}$.

a) If the point P(t) is located in the 3rd quadrant,

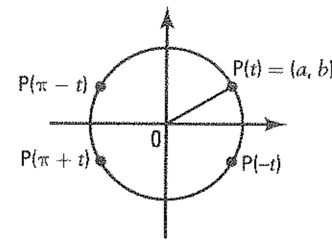
1. determine the x-coordinate $\cos t$. ***$\cos t = \frac{-3}{5}$***
2. deduce, in degrees, the value of t knowing that $180° \leqslant t \leqslant 270°$. ***$t = 233.1°$*** ✓

b) If the point P(t) is located in the 4th quadrant,

1. determine the x-coordinate $\cos t$. ***$\cos t = \frac{3}{5}$***
2. deduce, in degrees, the value of t knowing that $270° \leqslant t \leqslant 360°$. ***$t = 306.9°$*** ✓

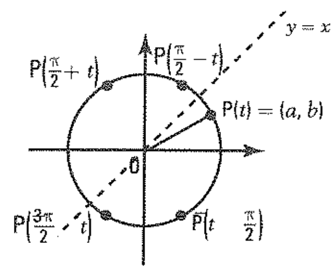
ACTIVITY 5 Properties of trigonometric points

a) Consider the trigonometric points represented on the unit circle on the right.



1. Name the geometric transformation that applies the point P(t) onto the trigonometric point
   1) P($\pi - t$) ***Reflection about the y-axis.***
   2) P($-t$) ***Reflection about the x-axis.***
   3) P($\pi + t$) ***Reflection about the y-axis, followed by a reflection about the x-axis.***
2. Deduce the Cartesian coordinates of the points
   1) P($\pi - t$) ***(−a, b)*** 2) P($-t$) ***(a, −b)*** 3) P($\pi + t$) ***(−a, −b)***

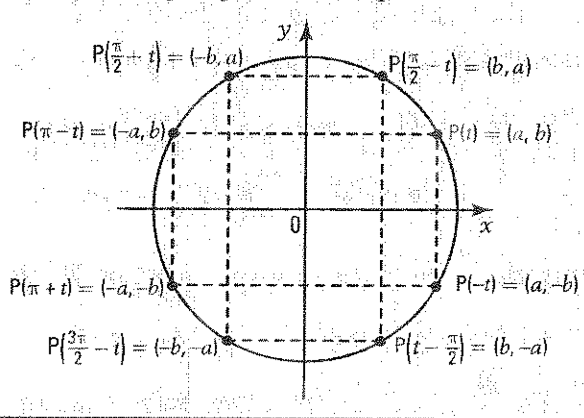
b) Consider the trigonometric points on the right.



1. Find the image of the point P(t) by the reflection about the line $y = x$, and deduce the coordinates of this image.
   $P\left(\frac{\pi}{2} - t\right) = (b, a)$
2. Deduce the coordinates of the trigonometric points
   1) $P\left(\frac{\pi}{2} + t\right)$ ***(−b, a)*** 2) $P\left(\frac{3\pi}{2} - t\right)$ ***(−b, −a)*** 3) $P\left(t - \frac{\pi}{2}\right)$ ***(b, −a)***

PROPERTIES OF TRIGONOMETRIC POINTS

Given a trigonometric point P(t) = (a, b), geometric transformations enable you to deduce the Cartesian coordinates of the following trigonometric points.



18. Consider the trigonometric point $P(t)=\left(\frac{5}{13}, \frac{12}{13}\right)$.

a) Deduce the coordinates of the following trigonometric points.

1. $P(-t)$ ***$\left(\frac{5}{13}, \frac{-12}{13}\right)$*** 2. $P(\pi - t)$ ***$\left(\frac{-5}{13}, \frac{12}{13}\right)$*** 3. $P(\pi + t)$ ***$\left(\frac{-5}{13}, \frac{-12}{13}\right)$***

b) Deduce

1. $\cos t$ ***$\frac{5}{13}$*** 2. $\sin t$ ***$\frac{12}{13}$*** 3. $\cos(-t)$ ***$\frac{5}{13}$***

4. $\sin(-t)$ ***$\frac{-12}{13}$*** 5. $\cos(\pi - t)$ ***$\frac{-5}{13}$*** 6. $\sin(\pi + t)$ ***$\frac{-12}{13}$***

19. In each of the following cases, choose the correct answer from the following 4 possible answers: $\cos t$, $\sin t$, $-\cos t$ and $-\sin t$.

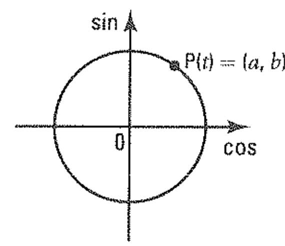
a) $\cos(-t)$ ***cos t*** **b)** $\sin(-t)$ ***–sin t*** **c)** $\cos(\pi + t)$ ***–cos t*** **d)** $\sin(\pi + t)$ ***–sin t***

e) $\cos(\pi + t)$ ***–cos t*** **f)** $\sin(\pi - t)$ ***sin t*** **g)** $\cos\left(\frac{\pi}{2} - t\right)$ ***sin t*** **h)** $\sin\left(\frac{\pi}{2} - t\right)$ ***cos t***

i) $\cos\left(\frac{\pi}{2} + t\right)$ ***–sin t*** **j)** $\sin\left(\frac{\pi}{2} + t\right)$ ***cos t*** **k)** $\cos\left(t - \frac{\pi}{2}\right)$ ***sin t*** **l)** $\sin\left(t - \frac{\pi}{2}\right)$ ***–cos t***

ACTIVITY 6 Periodicity of trigonometric points

Given the trigonometric point $P(t) = (a, b)$.



a) Determine the coordinates of the following trigonometric points.

1. $P(t + 2\pi)$ ***(a, b)*** 2. $P(t + 4\pi)$ ***(a, b)*** 3. $P(t + 6\pi)$ ***(a, b)***

4. $P(t - 2\pi)$ ***(a, b)*** 5. $P(t - 4\pi)$ ***(a, b)*** 6. $P(t - 6\pi)$ ***(a, b)***

b) 1. What can we say about the 6 trigonometric points defined in a)?
They are coincident.

2. What is the smallest positive real number a such that $P(t) = P(t + a)$? ***2π***

3. Is it true to say that $P(t) = P(t + 2\pi n)$, $n \in \mathbb{Z}$? ***True***

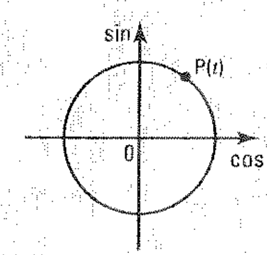
c) If $n \in \mathbb{Z}$, is it true to say that

1. $\cos t = \cos(t + 2\pi n)$? ***True*** 2. $\sin t = \sin(t + 2\pi n)$? ***True***

PERIODICITY OF TRIGONOMETRIC POINTS

- Given a trigonometric point $P(t)$. The smallest positive real number p such that $P(t + p) = P(t)$ is called the period.
This period is equal to 2π, that is

$$P(t + 2\pi) = P(t), \forall t \in \mathbb{R}$$



- Note that $P(t) = P(t + 2\pi n)$, $n \in \mathbb{Z}$.
Thus, $\ldots P(t - 4\pi) = P(t - 2\pi) = P(t) = P(t + 2\pi) = P(t + 4\pi) \ldots$
- Consequently, $\forall t \in \mathbb{R}, \forall n \in \mathbb{Z}$, we have:

$$\cos(t + 2\pi n) = \cos t \quad \text{and} \quad \sin(t + 2\pi n) = \sin t$$

Ex.: If $P(t) = (0.6; 0.8)$ then $P(t + 10\pi) = (0.6; 0.8)$.
$\cos(t + 10\pi) = 0.6$ and $\sin(t + 10\pi) = 0.8$.

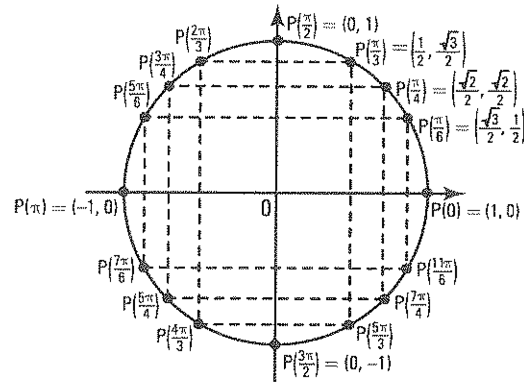
20. Write each of the following trigonometric points in the form P(t) where $0 \leq t < 2\pi$.

a) $P\left(\frac{13\pi}{6}\right)$ $P\left(\frac{\pi}{6}\right)$ b) $P(2\pi)$ $P(0)$ c) $P\left(\frac{17\pi}{4}\right)$ $P\left(\frac{\pi}{4}\right)$

d) $P\left(\frac{-5\pi}{3}\right)$ $P\left(\frac{\pi}{3}\right)$ e) $P\left(\frac{-10\pi}{3}\right)$ $P\left(\frac{2\pi}{3}\right)$ f) $P\left(\frac{121\pi}{6}\right)$ $P\left(\frac{\pi}{6}\right)$

ACTIVITY 7 Cartesian coordinates of the remarkable trigonometric points

The three remarkable trigonometric points of the 1st quadrant $P\left(\frac{\pi}{6}\right)$, $P\left(\frac{\pi}{4}\right)$ and $P\left(\frac{\pi}{3}\right)$ are represented on the right.



a) Locate on the unit circle the remarkable points of the 2nd quadrant and give the Cartesian coordinates of these points.

1. $P\left(\frac{2\pi}{3}\right)$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
2. $P\left(\frac{3\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. $P\left(\frac{5\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

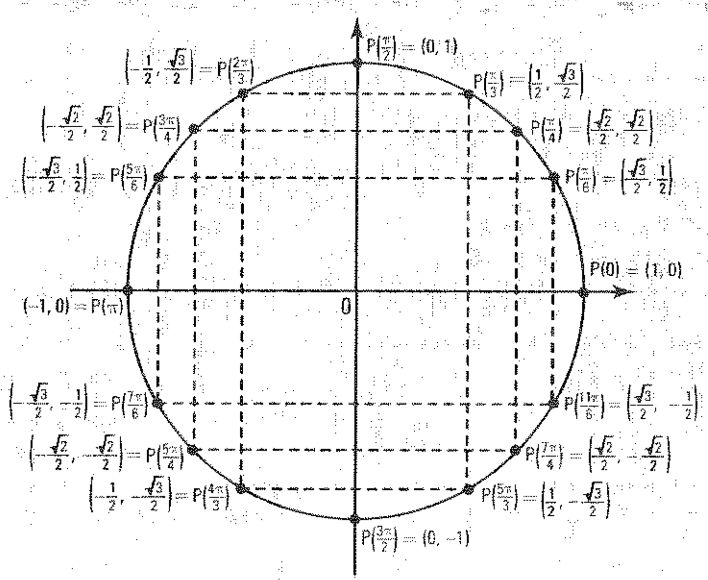
b) Locate on the trigonometric circle the remarkable points of the 3rd quadrant and give the Cartesian coordinates of these points.

1. $P\left(\frac{7\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ 2. $P\left(\frac{5\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ 3. $P\left(\frac{4\pi}{3}\right)$ $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

c) Locate on the trigonometric circle the remarkable points of the 4th quadrant and give the Cartesian coordinates of these points.

1. $P\left(\frac{5\pi}{3}\right)$ $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$ 2. $P\left(\frac{7\pi}{4}\right)$ $\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ 3. $P\left(\frac{11\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

REMARKABLE TRIGONOMETRIC POINTS



From the coordinates of the remarkable points of the 1st quadrant, we deduce by symmetry the coordinates of the remarkable points of the three other quadrants.

21. Determine the Cartesian coordinates of the following trigonometric points.

a) $P\left(-\frac{\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ b) $P\left(-\frac{3\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ c) $P\left(-\frac{4\pi}{3}\right)$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

d) $P\left(\frac{7\pi}{3}\right)$ $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ e) $P\left(\frac{31\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ f) $P\left(-\frac{13\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

22. From the Cartesian coordinates of the remarkable trigonometric points, determine the exact value of

a) $\sin\left(\frac{2\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$ b) $\cos\left(\frac{7\pi}{6}\right)$ $-\frac{\sqrt{3}}{2}$ c) $\sin\left(\frac{3\pi}{4}\right)$ $\frac{\sqrt{2}}{2}$

d) $\sin\left(\frac{3\pi}{2}\right)$ -1 e) $\cos\left(\frac{11\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$ f) $\cos\left(\frac{4\pi}{3}\right)$ $-\frac{1}{2}$

23. Determine the exact value of

a) $\sin\left(-\frac{\pi}{6}\right)$ $-\frac{1}{2}$ b) $\cos\left(-\frac{\pi}{3}\right)$ $\frac{1}{2}$ c) $\sin\left(\frac{13\pi}{6}\right)$ $\frac{1}{2}$

d) $\cos\left(\frac{19\pi}{4}\right)$ $-\frac{\sqrt{2}}{2}$ e) $\sin\left(-\frac{7\pi}{3}\right)$ $-\frac{\sqrt{3}}{2}$ f) $\cos\left(-\frac{17\pi}{4}\right)$ $\frac{\sqrt{2}}{2}$

24. Determine the exact value of

a) $\tan\left(\frac{2\pi}{3}\right)$ $-\sqrt{3}$ b) $\cot\left(\frac{5\pi}{6}\right)$ $-\sqrt{3}$

c) $\sec\left(\frac{7\pi}{6}\right)$ $-\frac{2\sqrt{3}}{3}$ d) $\csc\left(\frac{11\pi}{6}\right)$ -2

25. Knowing that $0 \leq t \leq 2\pi$, determine the two values of t such that

a) $\cos t = \frac{1}{2}$ $\frac{\pi}{3}$ *or* $\frac{5\pi}{3}$ b) $\sin t = \frac{\sqrt{3}}{2}$ $\frac{\pi}{3}$ *or* $\frac{2\pi}{3}$

c) $\cos t = \frac{-\sqrt{3}}{2}$ $\frac{5\pi}{6}$ *or* $\frac{7\pi}{6}$ d) $\sin t = -\frac{1}{2}$ $\frac{7\pi}{6}$ *or* $\frac{11\pi}{6}$

26. Find t if

a) $\sin t = \frac{1}{2}$ and $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$ $t = \frac{5\pi}{6}$ b) $\cos t = \frac{-1}{2}$ and $\pi \leq t \leq \frac{3\pi}{2}$ $t = \frac{4\pi}{3}$

c) $\sin t = \frac{-\sqrt{3}}{2}$ and $\frac{3\pi}{2} \leq t \leq 2\pi$ $t = \frac{5\pi}{3}$ d) $\cos t = \frac{1}{2}$ and $0 \leq t \leq \frac{\pi}{2}$ $t = \frac{\pi}{3}$

27. Knowing that $0 \leq t \leq 360°$, find the two values of t (to the nearest tenth) such that

a) $\cos t = 0.8$ ***36.9° or 323.1°*** b) $\cos t = -0.6$ ***126.9° or 233.1°***

c) $\sin t = 0.2$ ***11.5° or 168.5°*** d) $\sin t = -0.4$ ***203.6° or 336.4°***

28. Knowing that $0 \leq t \leq 2\pi$, find the two values of t (to the nearest hundredth) such that

a) $\sin t = 0.7$ ***0.78 rad or 2.37 rad*** b) $\sin t = -0.6$ ***5.64 rad or 3.79 rad***

c) $\cos t = 0.2$ ***1.37 rad or 4.91 rad*** d) $\cos t = -0.8$ ***2.50 rad or 3.79 rad***

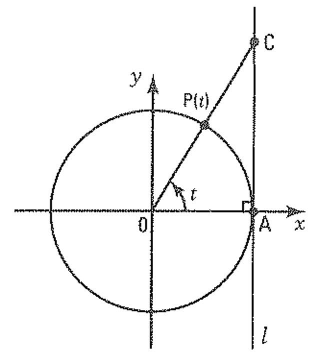
ACTIVITY 8 Trigonometric circle and the tangent axis

a) On the trigonometric circle on the right, line l is drawn perpendicular to the x-axis at the point A(1, 0).

Given a trigonometric point P(t) of the first quadrant, we designate C as the intersection of the extension of the radius 0P and the line l.

Explain why $\tan t = m\overline{AC}$.

In the right triangle 0AC, we have: $\tan t = \dfrac{m\overline{AC}}{m\overline{0A}} = \dfrac{m\overline{AC}}{1}$.

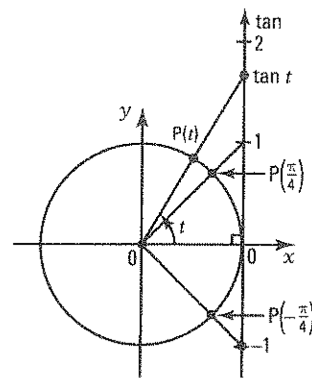


b) On the trigonometric circle on the right, we have directed and scaled the line l. The chosen scale corresponds to the radius of the unit circle. The resulting line l, scaled and directed, is called the **tangent axis**.

Given a trigonometric point P(t), we locate tan t on the tangent axis by extending the radius $\overline{0P}$ (see a)).

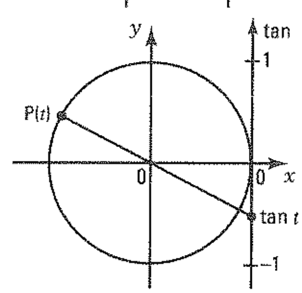
1. Locate the point $P\left(\frac{\pi}{4}\right)$ and verify by construction that $\tan \frac{\pi}{4} = 1$.

2. Locate the point $P\left(\frac{-\pi}{4}\right)$ and verify by construction that $\tan\left(-\frac{\pi}{4}\right) = -1$.



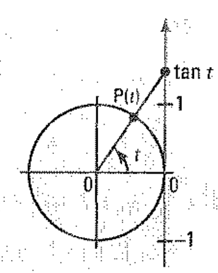
c) Given a trigonometric point P(t) in the 2nd quadrant, explain how to locate tan t on the tangent axis and deduce the sign of tan t.

We extend $\overline{0P}$. The intersection of the extension of $\overline{0P}$ and the tangent axis corresponds to tan t. We have: tan t < 0.



TRIGONOMETRIC CIRCLE AND THE TANGENT AXIS

- The trigonometric circle and **tangent** axis are represented on the right. Given a trigonometric point P(t), to locate tan t on the tangent axis, we extend the radius 0P. The intersection of the extension of the radius 0P and the tangent axis enables you to locate tan t.



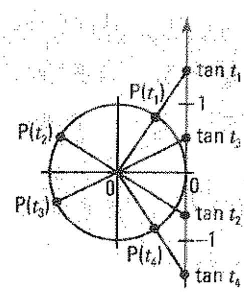
- Note that

$0 \leq t < \frac{\pi}{2} \Rightarrow \tan t \geq 0.$

$\frac{\pi}{2} < t \leq \pi \Rightarrow \tan t \leq 0.$

$\pi \leq t < \frac{3\pi}{2} \Rightarrow \tan t \geq 0.$

$\frac{3\pi}{2} < t \leq 2\pi \Rightarrow \tan t \leq 0.$



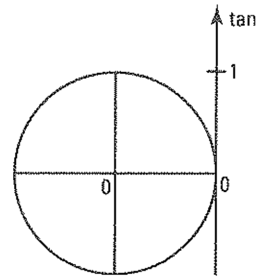
29. **a)** Consider the trigonometric point P(t).

Determine in which quadrant(s) P(t) is located if

1. $\tan t > 0$. ***In the 1st or 3rd quadrant.***
2. $\tan t < 0$. ***In the 2nd or 4th quadrant.***

b) If $0 \leqslant t \leqslant 2\pi$, determine t in each of the following cases.

1. $\tan t = 1$ $t = \frac{\pi}{4}$ **or** $t = \frac{5\pi}{4}$ 2. $\tan t = -1$ $t = \frac{3\pi}{4}$ **or** $t = \frac{7\pi}{4}$



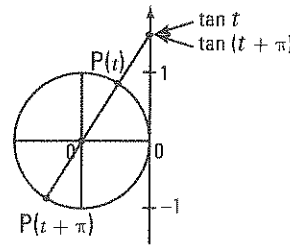
30. Consider the trigonometric circle on the right.

a) Randomly place a trigonometric point P(t).

b) Deduce the location of the trigonometric point P($t + \pi$).

c) Locate on the tangent axis tan t and tan ($t + \pi$) and compare tan t and tan ($t + \pi$). ***tan t = tan (t + π)***

d) Is it true to say that for any real t, we have: $\tan (t + \pi) = \tan t$? ***Yes***



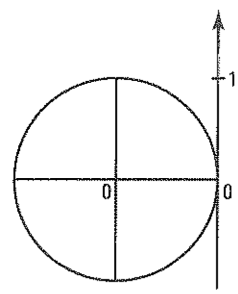
31. Consider the trigonometric point P(t).

If $0 \leqslant t \leqslant 2\pi$, determine the two possible solutions for t in each of the following cases.

a) $\tan t = \sqrt{3}$ $\frac{\pi}{3}$ **or** $\frac{4\pi}{3}$ **b)** $\tan t = -\sqrt{3}$ $\frac{2\pi}{3}$ **or** $\frac{5\pi}{3}$

c) $\tan t = \frac{\sqrt{3}}{3}$ $\frac{\pi}{6}$ **or** $\frac{7\pi}{6}$ **d)** $\tan t = \frac{-\sqrt{3}}{3}$ $\frac{5\pi}{6}$ **or** $\frac{11\pi}{6}$

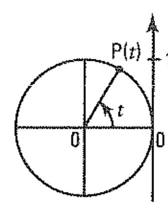
e) $\tan t = 0$ ***0 or π*** **f)** $\tan t = 0.7$ ***0.61 or 3.75***



32. **a)** The trigonometric circle and a trigonometric point P(t) in the 1st quadrant are represented on the right.

As t approaches $\frac{\pi}{2}$, what can we say about tan t?

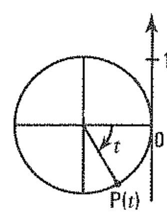
tan t increases indefinitely while remaining positive.



b) The trigonometric circle and a trigonometric point P(t) in the 4th quadrant are represented on the right.

As t approaches $\frac{-\pi}{2}$, what can we say about tan t?

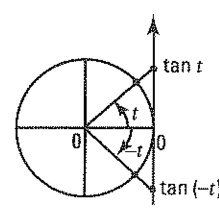
tan t decreases indefinitely while remaining negative.



33. **a)** Place the trigonometric points P(t) and P($-t$) on the unit circle.

b) Compare tan($-t$) and tan t. ***tan (−t) = −tan t.***

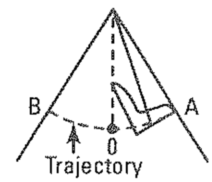
c) Is it true to say that for any real t, we have: $\tan (-t) = -\tan t$. ***Yes***



5.4 Periodic functions

ACTIVITY 1 Movement of a swing

Initially ($t = 0$), the seat of a swing is in position A. We notice that it takes 8 seconds for the swing to go back and forth 5 times.



a) Does the seat periodically follow the same trajectory? ***Yes***
If yes, each return is called a **cycle**.

b) The required time p to return to point A for the first time, ie the duration of one cycle, is called the **period** of the movement. Determine the period p of the movement.
p = 1.6 s

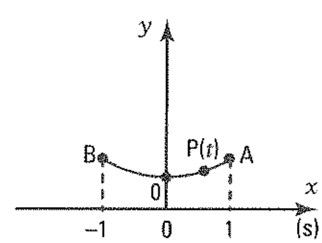
c) Complete the table below which gives the position P(t) (A, 0 or B) of the movement as a function of elapsed time t since the start.

t	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	4.8	6.4	8
P(t)	*A*	*0*	*B*	*0*	*A*	*0*	*B*	*0*	*A*	*A*	*A*	*A*

d) Verify that for any value of t ($0 \leqslant t \leqslant 8$), we have: $P(t) = P(t + p)$ where p is the period of the movement.
Various answers

e) The **frequency** of the movement is the number of cycles per unit of time. Given that we observe here 5 cycles in 8 seconds, calculate the frequency F, ie the number of cycles per second.
$F = \frac{5}{8} = 0.625$ ***cycles/sec***

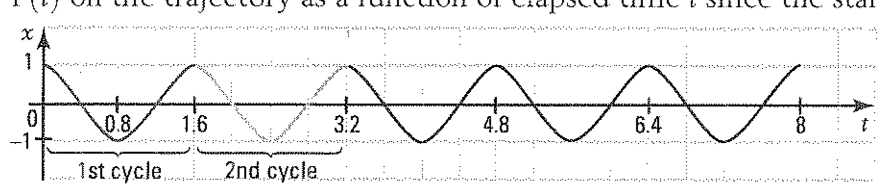
f) The trajectory of the swing's seat is represented by the arc AB in the Cartesian plane on the right. The points A, 0 and B have x-coordinates 1, 0 and –1 respectively.



1. In what interval is the x-coordinate of a point P(t) on the trajectory? $-1 \leqslant x \leqslant 1$
2. Complete the table of values below which gives the x-coordinate of a point P(t) on the trajectory as a function of elapsed time t since the start.

t	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	4.8	6.4	8
x	*1*	*0*	*–1*	*0*	*1*	*0*	*–1*	*0*	*1*	*1*	*1*	*1*

3. We have represented two cycles of the function f which gives the x-coordinate of a point P(t) on the trajectory as a function of elapsed time t since the start. Draw 3 other cycles.



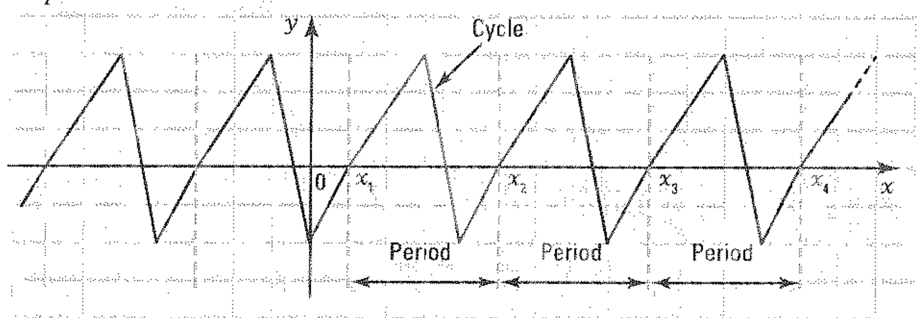
PERIODIC FUNCTIONS

- A function f is called **periodic** if there exists a real number a such that for any value x of the domain of f, we have:

$$f(x + a) = f(x)$$

The smallest positive value of a for which $f(x + a) = f(x)$ is called the **period** (denoted p) of the function.

- One **cycle** of a periodic function is the smallest portion of the graph that, by repetition, forms the curve representing the function. The distance between the extremities of the cycle is equal to the period p of the function.



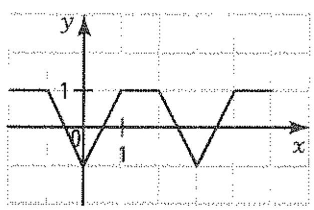
$p = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \ldots$

- The **frequency**, denoted F, is the reciprocal of the period. $F = \frac{1}{p}$

In a situation where the variable x represents time, the period represents the **duration of one cycle** and the frequency represents the **number of cycles** per unit of time.

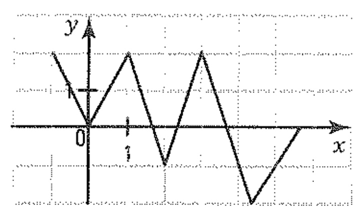
1. Indicate if the following functions are periodic. If yes, indicate the period p of the function.

a)



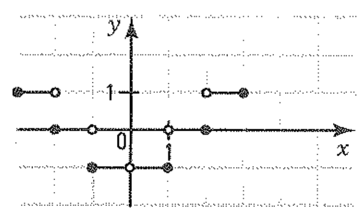
Yes , p = 3

b)



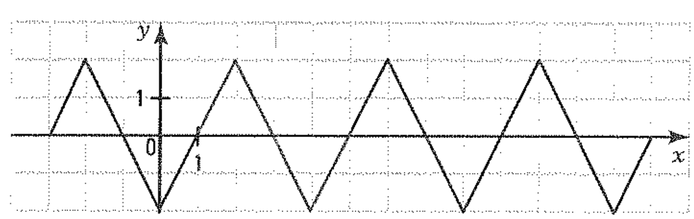
No

c)



No

2. One cycle of a periodic function f is represented on the right.



a) Determine the period p and the frequency F of this function.

p = 4, F = $\frac{1}{4}$

b) Complete the graph of this function for $x \in [-3, 13]$.

c) Determine

1. ran f. ***[-2, 2]*** 2. min f. ***-2*** 3. max f. ***2***

d) Determine

1. $f(15)$. ***0*** 2. $f(25)$. ***0*** 3. $f(42)$. ***2*** 4. $f(32)$. ***-2***

e) What are the zeros of f over the interval $[0, 10]$? ***1, 3, 5, 7 and 9***

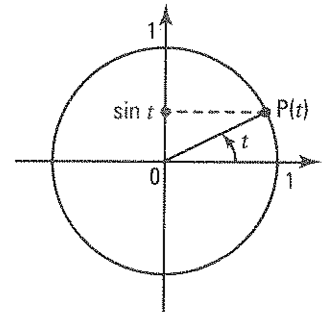
5.5 Sine function

ACTIVITY 1 Basic sine function

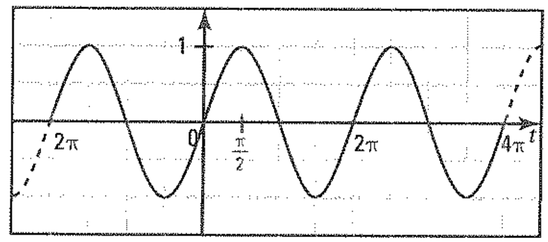
The basic sine function, denoted sin, has the rule $y = \sin t$.

a) Complete the table of values below when t varies from -2π to 4π.

t	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin t	*0*	*1*	*0*	*−1*	*0*	*1*	*0*	*−1*	*0*	*1*	*0*	*−1*	*0*



b) We have represented the function $y = \sin t$ on the right. Is this function periodic? If yes, what is the period p of the function? ***Yes, p = 2π***



c) For this function, determine

1. the domain. $\mathbb{R}$
2. the range. ***[−1, 1]***
3. the maximum. ***1***
4. the minimum. ***−1***

d) The amplitude A of a function f is equal to half the difference between the maximum and the minimum of the function, i.e. $A = \frac{\max f - \min f}{2}$.

What is the amplitude A of the sine function? ***1***

e) When $t \in [0, 2\pi]$, determine, for the sine function,

1. the zeros. ***0, π and 2π***
2. the sign. $\sin t \geq 0$ if $0 \leq t \leq \pi$ and $\sin t \leq 0$ if $\pi \leq t \leq 2\pi$
3. the variation. $\sin \nearrow$ if $t \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; $\sin \searrow$ if $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

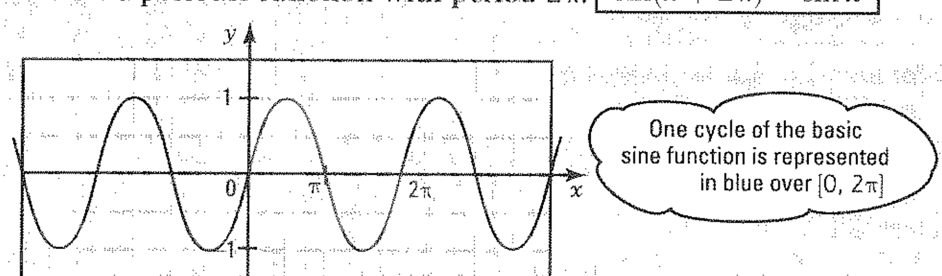
f) Verify the property $\forall\, t \in \mathbb{R}: \sin(-t) = -\sin t$, using the graph or trigonometric circle.

BASIC SINE FUNCTION

- The sine function, denoted sin, is defined by

$$\sin: \mathbb{R} \rightarrow \mathbb{R}$$
$$x \mapsto y = \sin x$$

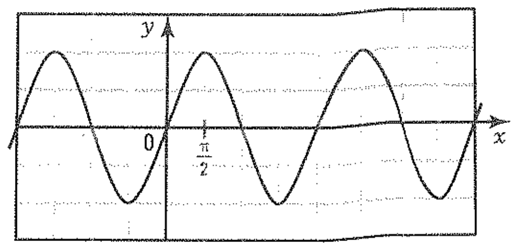
- The sine function is a periodic function with period 2π. $\boxed{\sin(x + 2\pi) = \sin x}$



- The **amplitude** of a sine function f is given by $A = \frac{\max f - \min f}{2}$.

  The amplitude of the basic sine function is: $A = 1$.
- We have:
  - domain $= \mathbb{R}$, range $= [-1, 1]$.
  - zeros over $[0, 2\pi]$: 0, π and 2π.
  - sign over $[0, 2\pi]$: $\sin x \geqslant 0$ if $x \in [0, \pi]$ and $\sin x \leqslant 0$ if $x \in [\pi, 2\pi]$.
  - variation over $[0, 2\pi]$: sin ↗ if $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; sin ↘ if $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
  - extrema: max $= 1$; min $= -1$.
- For any real x, we have: $\sin(-x) = -\sin x$. The basic sine function is therefore considered to be an **odd** function.

1. Consider the function $f(x) = \sin x$.



a) Find the zeros of f when

1. $x \in [-2\pi, 4\pi]$. ***-2π, $-\pi$, 0, π, 2π, 3π and 4π***
2. $x \in \mathbb{R}$. ***$\{\pi n\}$, $n \in \mathbb{Z}$***

b) Solve the inequality $\sin x \geqslant 0$ when

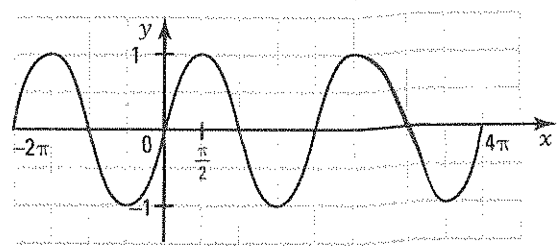
1. $x \in [-2\pi, 4\pi]$. ***$S = [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi]$***
2. $x \in \mathbb{R}$. ***$S = [2\pi n, \pi + 2\pi n]$, $n \in \mathbb{Z}$***

c) Find the values of x for which the function f is increasing when

1. $x \in [-2\pi, 4\pi]$. ***$\left[-2\pi, -\frac{3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \cup \left[\frac{7\pi}{2}, 4\pi\right]$***
2. $x \in \mathbb{R}$. ***$\left[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right]$, $n \in \mathbb{Z}$***

ACTIVITY 2 Equation $\sin \theta = k$

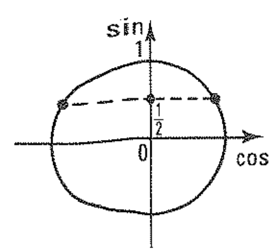
The function $y = \sin x$ is represented on the right when $x \in [-2\pi, 4\pi]$.



a) 1. By referring to the trigonometric circle, solve the equation $\sin \theta = \frac{1}{2}$ when $\theta \in [0, 2\pi]$.

$S = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

2. Explain how to find, from the solutions in 1, the solutions to the equation $\sin \theta = \frac{1}{2}$ when



1) $\theta \in [2\pi, 4\pi]$. ***Add the period 2π to each solution.***

2) $\theta \in [-2\pi, 0]$. ***Subtract the period 2π to each solution.***

3. Verify that the solution set S to the equation $\sin \theta = \frac{1}{2}$ over $\mathbb{R}$ is described by

$$S = \left\{\ldots, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots\right\} \text{ (enumeration).}$$

or by

$$S = \left\{\frac{\pi}{6} + 2\pi n\right\} \cup \left\{\frac{5\pi}{6} + 2\pi n\right\} \text{ where } n \in \mathbb{Z} \text{ (set-builder notation).}$$

b) By referring to the trigonometric circle and using a calculator, solve the equation $\sin\theta = 0.4$ when

1. $\theta \in [0, 2\pi]$. ***S = {0.41; 2.73}***
2. $\theta \in [4\pi, 6\pi]$. ***S = {12.98; 15.29}***
3. $\theta \in \mathbb{R}$. $S = \{0.41 + 2\pi n\} \cup \{2.73 + 2\pi n\},\ n \in \mathbb{Z}$

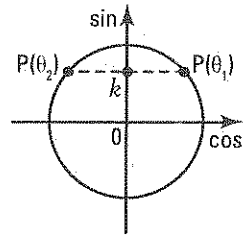
c) Solve the inequality $\sin\theta \geq \frac{1}{2}$ when

1. $\theta \in [0, 2\pi]$. $S = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
2. $\theta \in [0, 4\pi]$. $S = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup \left[\frac{13\pi}{6}, \frac{17\pi}{6}\right]$
3. $\theta \in \mathbb{R}$. $S = \left[\frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n\right],\ n \in \mathbb{Z}$

EQUATION $\sin\theta = k,\quad -1 \leq k \leq 1$

- When $\theta \in [0, 2\pi[$, the equation $\sin\theta = k$ yields 2 solutions θ_1 and θ_2.

$$\theta_1 = \sin^{-1} k \text{ and } \theta_2 = \pi - \theta_1$$



- When $\theta \in \mathbb{R}$, the equation $\sin\theta = k$ yields an infinite number of solutions.

$$S = \{\theta_1 + 2\pi n\} \cup \{\theta_2 + 2\pi n\} \text{ where } n \in \mathbb{Z}$$

Ex.: The equation $\sin\theta = \frac{\sqrt{2}}{2}$ has the solution set:

- $S = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ when $\theta \in [0, 2\pi[$
- $S = \left\{\frac{\pi}{4} + 2\pi n\right\} \cup \left\{\frac{3\pi}{4} + 2\pi n\right\},\ n \in \mathbb{Z}$ when $\theta \in \mathbb{R}$.

2. Solve the following equations over

1. $[0, 2\pi]$ 2. $[2\pi, 4\pi]$ 3. $\mathbb{R}$

a) $\sin x = \frac{\sqrt{3}}{2}$
1. $S = \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$
2. $S = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
3. $S = \left\{\frac{\pi}{3} + 2\pi n\right\} \cup \left\{\frac{2\pi}{3} + 2\pi n\right\},\ n \in \mathbb{Z}$

b) $\sin x = 1$
1. $S = \left\{\frac{\pi}{2}\right\}$
2. $S = \left\{\frac{5\pi}{2}\right\}$
3. $S = \left\{\frac{\pi}{2} + 2\pi n\right\}\ n \in \mathbb{Z}$

c) $\sin x = -\frac{1}{2}$
1. $S = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
2. $S = \left\{\frac{19\pi}{6}, \frac{23\pi}{6}\right\}$
3. $S = \left\{\frac{7\pi}{6} + 2\pi n\right\} \cup \left\{\frac{11\pi}{6} + 2\pi n\right\},\ n \in \mathbb{Z}$

d) $\sin x = 0.6$
1. ***S = {0.64; 2.5}***
2. ***S = {6.92; 8.78}***
3. $S = \{0.64 + 2\pi n\} \cup \{2.5 + 2\pi n\},\ n \in \mathbb{Z}$

3. Solve the following inequalities over

1. $[0, 2\pi]$ 2. $\mathbb{R}$

a) $\sin x \geq \frac{\sqrt{2}}{2}$
1. $S = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
2. $S = \left[\frac{\pi}{4} + 2\pi n, \frac{5\pi}{4} + 2\pi n\right],\ n \in \mathbb{Z}$

b) $\sin x \leq \frac{1}{2}$
1. $S = \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$
2. $S = \left[2\pi n, \frac{\pi}{6} + 2\pi n\right] \cup \left[\frac{5\pi}{6} + 2\pi n, 2\pi + 2\pi n\right],\ n \in \mathbb{Z}$

c) $\sin x \leqslant -\frac{1}{2}$ 1. $S = \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$ 2. $S = \left[\frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n\right]$

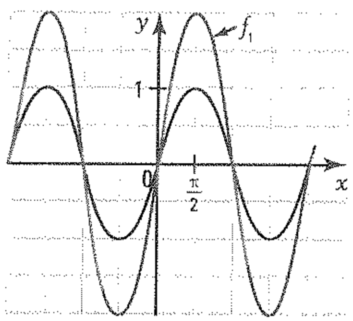
d) $\sin x \leqslant 1$ 1. $S = [0, 2\pi]$ 2. $S = \mathbb{R}$

ACTIVITY 3 Function $y = a \sin x$

a) For each of the following functions,

1. determine the value of parameter a.
2. indicate how to obtain the graph of the function from the graph of $y = \sin x$.
3. determine the amplitude.

$f_1(x) = 2 \sin x$

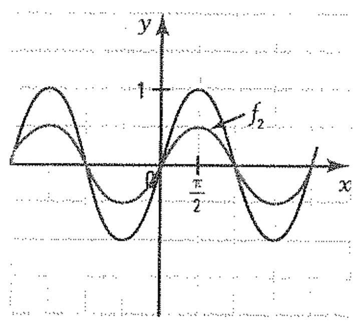


1. $a = 2$

2. Vertical stretch

3. Amplitude = 2

$f_2(x) = \frac{1}{2} \sin x$

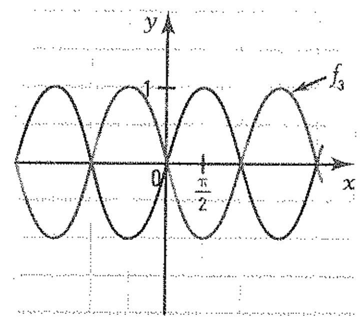


1. $a = \frac{1}{2}$

2. Vertical reduction

3. Amplitude = $\frac{1}{2}$

$f_3(x) = -\sin x$



1. $a = -1$

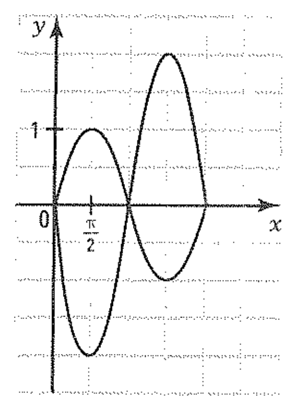
2. Reflection about the x-axis

3. Amplitude = 1

b) 1. From the graph of the function $y = \sin x$ $(0 \leqslant x \leqslant 2\pi)$, deduce the graph of $y = -2 \sin x$. Explain the procedure.

1. We perform a vertical stretch of factor 2.

2. We perform a reflection about the x-axis of the resulting figure.



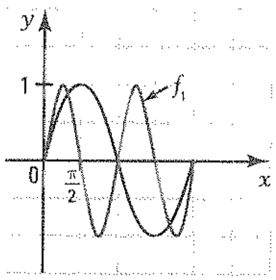
2. What is the amplitude of the function $y = -2 \sin x$? *2*

c) What is the relationship between the parameter a of the function $y = -2 \sin x$ and the amplitude of the function? *Amplitude = $|a|$*

ACTIVITY 4 Function $y = \sin bx$

a) For each of the following functions,
1. determine the value of parameter b.
2. indicate how to obtain the graph of the function from the graph of $y = \sin x$.
3. determine the period.

$f_1(x) = \sin 2x$

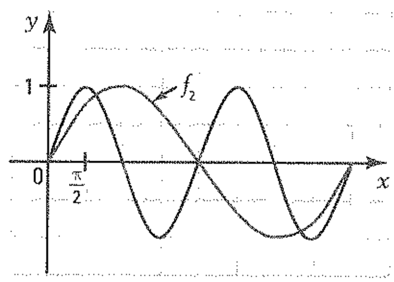


1. b = 2

2. Horizontal reduction

3. Period = π

$f_2(x) = \sin \frac{1}{2}x$

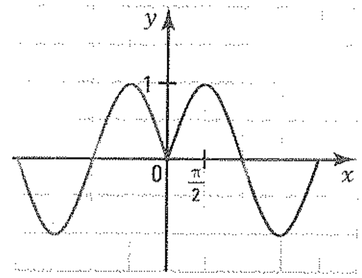


1. $b = \frac{1}{2}$

2. Horizontal stretch

3. Period = 4π

$f_3(x) = \sin(-x)$



1. b = −1

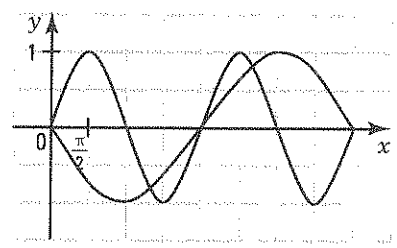
2. Reflection about the y-axis

3. Period = 2π

b) 1. From the graph of the function $y = \sin x$ $(0 \leqslant x \leqslant 4\pi)$, deduce the graph of $y = \sin\left(-\frac{1}{2}x\right)$. Explain the procedure.

1. We perform a horizontal stretch of factor 2.

2. We perform a reflection about the x-axis of the resulting figure.



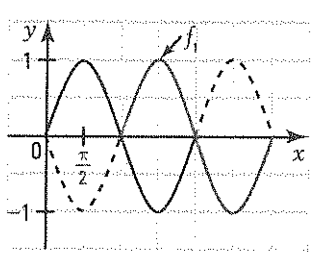
2. What is the period of the function $y = \sin\left(-\frac{1}{2}x\right)$? ***4π***

c) What is the relationship between the parameter b of the function $y = \sin bx$ and the period of the function? ***Period*** $= \frac{2\pi}{|b|}$

ACTIVITY 5 Function $y = \sin(x - h)$

a) For each of the following functions,
1. determine the value of parameter h.
2. indicate how to obtain the graph of the function from the graph of $y = \sin x$.

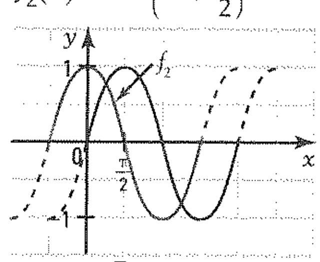
$f_1(x) = \sin(x - \pi)$



1. h = π

2. Horizontal translation of π units to the right.

$f_2(x) = \sin\left(x + \frac{\pi}{2}\right)$

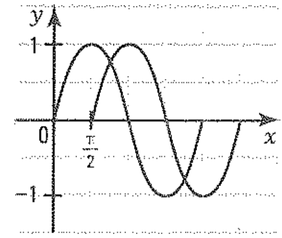


1. $h = -\frac{\pi}{2}$

2. Horizontal translation of $\frac{\pi}{2}$ ***units to the left.***

b) From the graph of the function $y = \sin x$ $(0 \leqslant x \leqslant 2\pi)$, deduce the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$. Explain the procedure.

We perform a horizontal translation of $\frac{\pi}{2}$ units to the right.

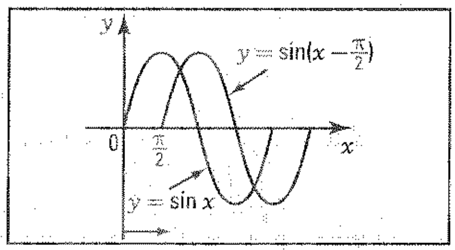


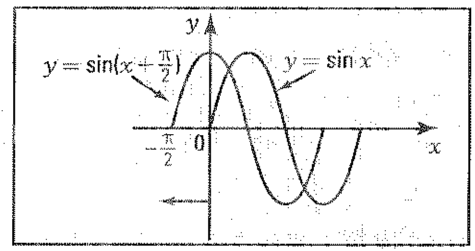
PHASE SHIFT

The graph of the function $y = \sin(x - h)$ is deduced from the graph of the function $y = \sin x$ by a horizontal translation of

- $|h|$ units to the right if $h > 0$.
- $|h|$ units to the left if $h < 0$.

Such a shift of the basic function is called the phase shift.





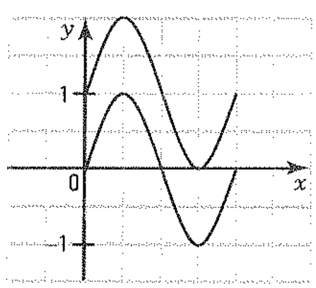
– The curve $y = \sin x$ is phase shifted $|h|$ units to the right.

– The curve $y = \sin x$ is phase shifted $|h|$ units to the left.

ACTIVITY 6 Function $y = \sin x + k$

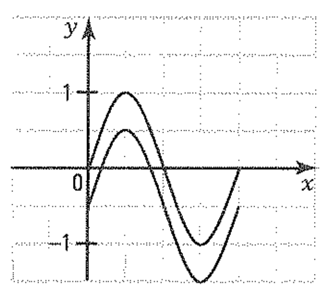
From the graph of $y = \sin x$ $(0 \leqslant x \leqslant 2\pi)$, deduce, explaining the procedure, the graph of

a) $y = \sin x + 1$.



Perform a vertical translation of 1 unit upward.

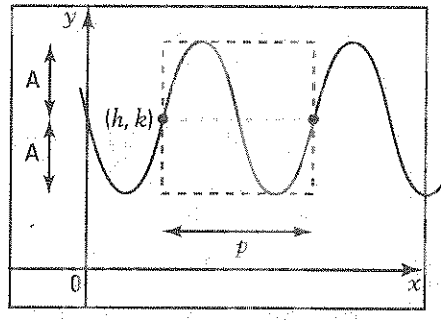
b) $y = \sin x - \frac{1}{2}$.



Perform a vertical translation of 0.5 units downward.

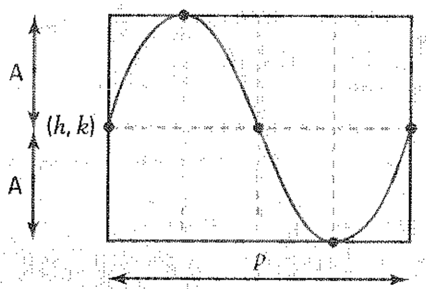
SINUSOIDAL FUNCTION $y = a \sin b(x - h) + k$

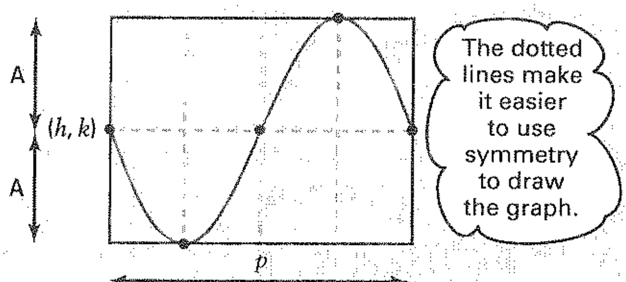
- The graph of the function $y = a \sin b(x - h) + k$ is deduced from the graph of the basic function $y = \sin x$ by the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$.
- We have:
  period: $p = \frac{2\pi}{|b|}$; amplitude: $A = |a|$;
  domain $= \mathbb{R}$;
  range $= [k - A, k + A]$.



- To draw one cycle of the graph of $y = a \sin b(x - h) + k$,
  1. we identify the point (h, k), starting point of one cycle.
  2. we draw a rectangle with a height equal to 2A, twice the amplitude, and a length equal to the period p.
  3. we draw one cycle inside the rectangle.
     There are two situations depending on the signs of a and b.

$ab > 0$	$ab < 0$
Increasing cycle from the start.	Decreasing cycle from the start.

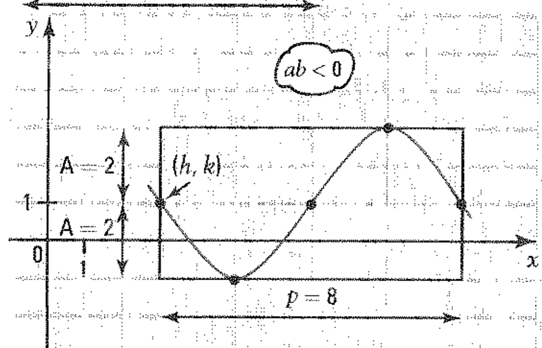




Ex.: Draw one cycle of the function
$y = -2 \sin \frac{\pi}{4}(x - 3) + 1$.

We have: $a = -2$, $b = \frac{\pi}{4}$, $h = 3$, $k = 1$

- starting point of the cycle: $(h, k) = (3,1)$.
- amplitude: $A = |a| = 2$.
- period: $p = \frac{2\pi}{|b|} = 8$.

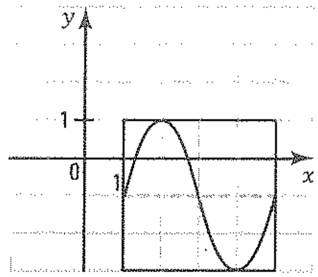


4. Consider the functions $f(x) = a \sin b(x - h) + k$ and $g(x) = -a \sin -b(x - h) + k$.
Explain why the functions f and g are equal.

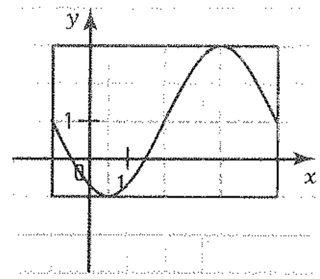
$-a \sin -b(x - h) + k = -a[-\sin b(x - h)] + k = a \sin b(x - h) + k$ since $\sin(-t) = -\sin t$.

5. Draw one cycle of each of the following functions.

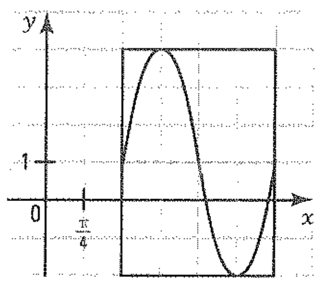
a) $y = 2\sin\frac{\pi}{2}(x-1) - 1$



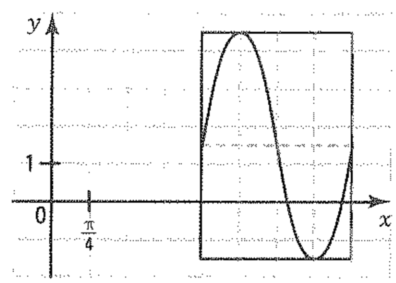
b) $y = -2\sin\frac{\pi}{3}(x+1) + 1$



c) $y = 3\sin 2\left(x - \frac{\pi}{2}\right) + 1$

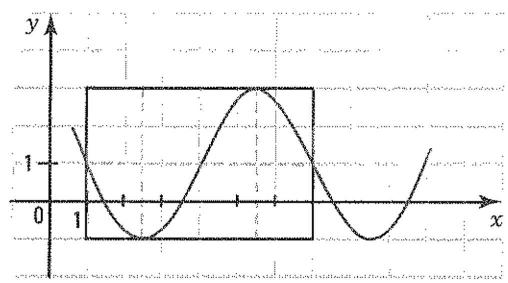


d) $y = -3\sin -2(x-\pi) + \frac{3}{2}$



ACTIVITY 7 Finding the zeros of the function $y = a\sin b(x-h) + k$

A portion of the graph of the function $f(x) = -2\sin\frac{\pi}{3}(x-1) + 1$ is represented on the right.



a) What is the period p of the function? ***p = 6***

b) 1. How many zeros does the function f have? ***An infinite number.***

2. How many zeros does the function f have when $x \in [1, 7]$? ***Two zeros.***

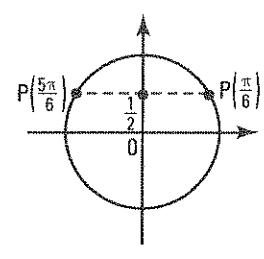
c) Justify the steps for finding the zeros of f when $x \in [1, 7]$.

1. $-2\sin\frac{\pi}{3}(x-1) + 1 = 0$ — ***We set f(x) = 0.***

2. $\sin\frac{\pi}{3}(x-1) = \frac{1}{2}$ — ***We isolate sin b(x − h).***

3. $\frac{\pi}{3}(x-1) = \frac{\pi}{6}$ or $\frac{\pi}{3}(x-1) = \frac{5\pi}{6}$ — ***We solve the equation*** $\sin\theta = \frac{1}{2}$ ***knowing that*** $\theta = \frac{\pi}{3}(x-1)$.

4. $x = 1.5$ or $x = 3.5$ — ***We deduce the solutions for x.***



d) From the zeros of f obtained over $[1, 7]$, explain how to deduce the zeros of f located on the next cycle, i.e. when $x \in [7, 13]$.

You need to add the period to each zero. The zeros over [7, 13] are 7.5 and 9.5.

e) Verify that the set of zeros of f is $\{\ldots; -4.5; -2.5; 1.5; 3.5; 7.5; 9.5; \ldots\}$ or $\{1.5 + 6n\} \cup \{3.5 + 6n\}, n \in \mathbb{Z}$.

FINDING THE ZEROS OF THE FUNCTION $f(x) = a \sin b(x - h) + k$

To determine the zeros of $f(x) = a \sin b(x - h) + k$,	Ex.: $f(x) = 2 \sin \frac{\pi}{2}(x - 1) - 1$
1. we establish the period of the function. $p = \frac{2\pi}{\lvert b \rvert}$	$p = 4$
2. we solve the equation $f(x) = 0$.	$2 \sin \frac{\pi}{2}(x - 1) - 1 = 0$
– Isolate $\sin b(x - h)$.	$\sin \frac{\pi}{2}(x - 1) = \frac{1}{2}$
– Find the angles $b(x - h)$ over the interval $[0, 2\pi]$ that verify the equation.	$\frac{\pi}{2}(x - 1) = \frac{\pi}{6}$ or $\frac{\pi}{2}(x - 1) = \frac{5\pi}{6}$
– Isolate x.	$x = \frac{4}{3}$ or $x = \frac{8}{3}$
– Determine the solution set, taking the period into consideration.	$S = \left\{\frac{4}{3} + 4n\right\} \cup \left\{\frac{8}{3} + 4n\right\}$

6. The following functions have the rule $f(x) = a \sin b(x - h) + k$.
Find the zeros of each function over
1. the interval $[h, h + p]$ where p is the period of the function.
2. the set of all real numbers.

a) $f(x) = 2 \sin \frac{\pi}{6}(x - 2) + 1$

$\sin \frac{\pi}{6}(x - 2) = \frac{-1}{2}$

$\frac{\pi}{6}(x - 2) = \frac{7\pi}{6}$ or $\frac{\pi}{6}(x - 2) = \frac{11\pi}{6}$

$x = 9$ or $x = 13$

1. $S = \{9, 13\}$ over $[2, 14]$

2. $S = \{9 + 12n\} \cup \{13 + 12n\}$

b) $f(x) = -2 \sin \frac{\pi}{3}(x + 1) + \frac{1}{2}$

$\sin \frac{\pi}{3}(x + 1) = 0.25$

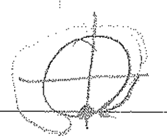
$\frac{\pi}{3}(x + 1) = 0.25$ or $\frac{\pi}{3}(x + 1) = 2.89$

$x = -0.76$ or $x = 1.76$

1. $S = \{-0.76; 1.76\}$ over $[-1, 5]$

2. $S = \{-0.76 + 6n\} \cup \{1.76 + 6n\}$

c) $f(x) = \sin 2(x - \pi) + 1$



$\sin 2(x - \pi) = -1$

$2(x - \pi) = \frac{3\pi}{2}$

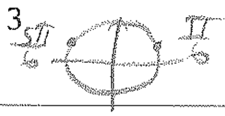
$x = \frac{7\pi}{4}$

h, h+p

1. $S = \left\{\frac{7\pi}{4}\right\}$ over $[\pi, 2\pi]$

2. $S = \left\{\frac{7\pi}{4} + \pi n\right\}$

d) $f(x) = 6 \sin\left(x + \frac{\pi}{2}\right) - 3$



$\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$

$x + \frac{\pi}{2} = \frac{\pi}{6}$ or $x + \frac{\pi}{2} = \frac{5\pi}{6}$

$x = -\frac{\pi}{3}$ or $x = \frac{\pi}{3}$

1. $S = \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$ over $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

2. $S = \left\{-\frac{\pi}{3} + 2\pi n\right\} \cup \left\{\frac{\pi}{3} + 2\pi n\right\}$

e) $f(x) = -3 \sin \frac{\pi}{4}x + 6$

$\sin \frac{\pi}{4}x = 2$

1. $S = \varnothing$

2. $S = \varnothing$

f) $f(x) = -2 \sin \frac{\pi}{8}(x + 2) + \sqrt{2}$

$\sin \frac{\pi}{8}(x + 2) = \frac{\sqrt{2}}{2}$

$\frac{\pi}{8}(x + 2) = \frac{\pi}{4}$ or $\frac{\pi}{8}(x + 2) = \frac{3\pi}{4}$

1. $S = \{0, 4\}$ over $[-2, 14]$

2. $S = \{0 + 16n\} \cup \{4 + 16n\}$

7. Determine the zeros of the function $f(x) = -2\sin\frac{\pi}{12}(x+5) - 1$ over the interval $[90, 150]$.

$\sin\frac{\pi}{12}(x+5) = -\frac{1}{2}$

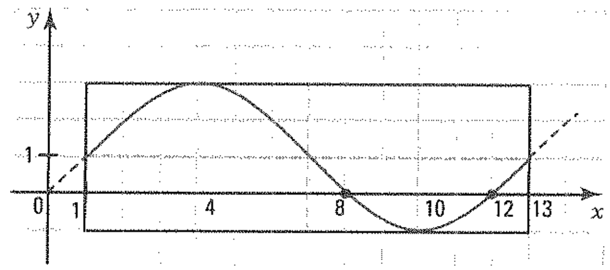
$\frac{\pi}{12}(x+5) = \frac{7\pi}{6}$ or $\frac{\pi}{12}(x+5) = \frac{11\pi}{6}$

$x = 9$ or $x = 17$

The zeros are: 105, 113, 129, 137.

ACTIVITY 8 Study of the function $f(x) = a\sin b(x-h) + k$

One cycle of the function $f(x) = 2\sin\frac{\pi}{6}(x-1) + 1$ is represented below.



a) Determine

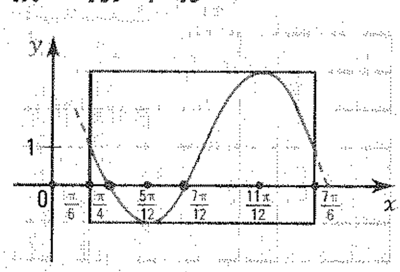
1. the period. *12*
2. the amplitude. *2*

b) Determine

1. dom f. $\mathbb{R}$
2. ran f. *[-1, 3]*
3. the zeros of f over $[1, 13]$. *8 and 12*
4. the zeros of f over $\mathbb{R}$. $\{8 + 12n\} \cup \{12 + 12n\}$
5. the sign of f over $[1, 13]$. $f(x) \geq 0$ over $[1, 8] \cup [12, 13]$ and $f(x) \leq 0$ over $[8, 12]$
6. the sign of f over $\mathbb{R}$. $f(x) \geq 0$ over $[1 + 12n, 8 + 12n] \cup [12 + 12n, 13 + 12n]$
   $f(x) \leq 0$ over $[8 + 12n, 12 + 12n]$
7. the variation of f over $[1, 13]$. $f \nearrow$ over $[1, 4] \cup [10, 13]$ and $f \searrow$ over $[4, 10]$
8. the variation of f over $\mathbb{R}$. $f \nearrow$ over $[1 + 12n, 4 + 12n] \cup [10 + 12n, 13 + 12n]$
   $f \searrow$ over $[4 + 12n, 10 + 12n]$
9. the maximum and minimum of f. max $f = 3$; min $f = -1$

STUDY OF THE FUNCTION $f(x) = a\sin b(x-h) + k$

Given $f(x) = -2\sin 2\left(x - \frac{\pi}{6}\right) + 1$. We have:



- dom $f = \mathbb{R}$; ran $f = [-1, 3]$
- period $p = \pi$; amplitude $A = 2$
- zeros of f: $\left\{\frac{\pi}{4} + \pi n\right\} \cup \left\{\frac{7\pi}{12} + \pi n\right\}$
- sign of f:

  $f(x) \geq 0$ over $\left[\frac{\pi}{6} + \pi n, \frac{\pi}{4} + \pi n\right] \cup \left[\frac{7\pi}{12} + \pi n, \frac{7\pi}{6} + \pi n\right]$

  $f(x) \leq 0$ over $\left[\frac{\pi}{4} + \pi n, \frac{7\pi}{12} + \pi n\right]$
- variation of f:

  $f \searrow$ over $\left[\frac{\pi}{6} + \pi n, \frac{5\pi}{12} + \pi n\right] \cup \left[\frac{11\pi}{12} + \pi n, \frac{7\pi}{6} + \pi n\right]$

  $f \nearrow$ over $\left[\frac{5\pi}{12} + \pi n, \frac{11\pi}{12} + \pi n\right]$
- min $f = -1$; max $f = 3$.

8. For each of the following functions, determine
1. the period, 2. the amplitude, 3. the range of the function.

a) $f(x) = -2 \sin \frac{\pi}{8}(x-5)+3$

1. $p = 16$
2. $A = 2$
3. $\text{ran } f = [1, 5]$

b) $f(x) = 3 \sin 12\left(x+\frac{\pi}{2}\right)+5$

1. $p = \frac{\pi}{6}$
2. $A = 3$
3. $\text{ran } f = [2, 8]$

c) $f(x) = 5 \sin \frac{4\pi}{3}(x+1)-4$

1. $p = \frac{3}{2}$
2. $A = 5$
3. $\text{ran } f = [-9, 1]$

d) $f(x) = 10 \sin \frac{6}{5}\left(x-\frac{\pi}{4}\right)+4$

1. $p = \frac{5\pi}{3}$
2. $A = 10$
3. $\text{ran } f = [-6, 14]$

9. Determine the initial value of the following functions.

a) $f(x) = 4 \sin \frac{\pi}{6}(x+1)-3$ -1

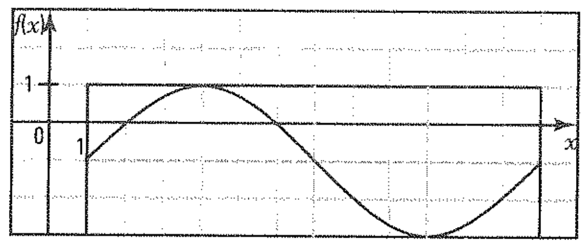
b) $f(x) = -2 \sin \frac{\pi}{3}(x-2)+2$ $\sqrt{3}+2$

c) $f(x) = 2 \sin 2\left(x-\frac{\pi}{4}\right)+4$ 2

d) $f(x) = 3 \sin \pi(x+5) - 1$ -1

10. For each of the following functions, determine, over $\mathbb{R}$, the interval over which the function is positive.

a) $f(x) = 2 \sin \frac{\pi}{6}(x-1)-1$



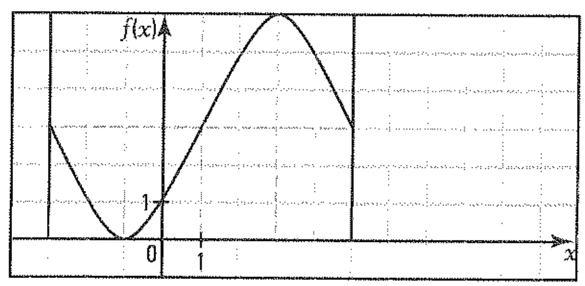
Zeros: $\sin \frac{\pi}{6}(x-1) = \frac{1}{2}$

$\frac{\pi}{6}(x-1) = \frac{\pi}{6}$ or $\frac{\pi}{6}(x-1) = \frac{5\pi}{6}$

$x = 2$ $\quad$ $x = 6$

$f(x) \geq 0$ over $[2 + 12n, 6 + 12n]$

b) $f(x) = -3 \sin \frac{\pi}{4}(x+3)+3$



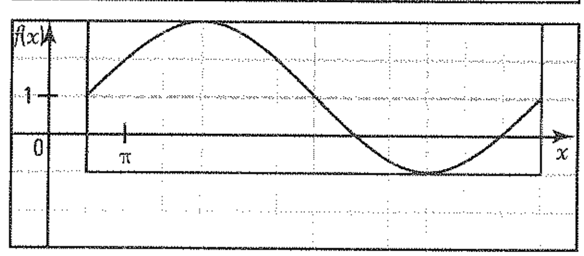
Zeros: $\sin \frac{\pi}{4}(x+3) = 1$

$\frac{\pi}{4}(x+3) = \frac{\pi}{2}$

$x = -1$

$f(x) \geq 0$ over $\mathbb{R}$

c) $f(x) = 2 \sin \frac{1}{3}\left(x-\frac{\pi}{2}\right)+1$



Zeros: $\sin \frac{1}{3}\left(x-\frac{\pi}{2}\right) = -\frac{1}{2}$

$\frac{1}{3}\left(x-\frac{\pi}{2}\right) = \frac{7\pi}{6}$ or $\frac{1}{3}\left(x-\frac{\pi}{2}\right) = \frac{11\pi}{6}$

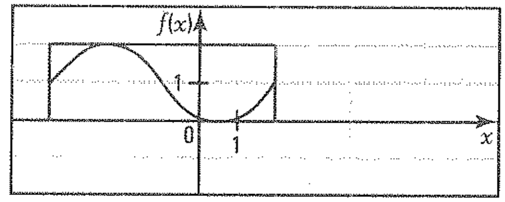
$x = 4\pi$ $\quad$ $x = 6\pi$

$f(x) \geq 0$ over $\left[\frac{\pi}{2} + 6\pi n, 4\pi + 6\pi n\right] \cup \left[6\pi + 6\pi n, \frac{13\pi}{2} + 6\pi n\right]$

11. For each of the following functions, determine the interval over which the function is decreasing.

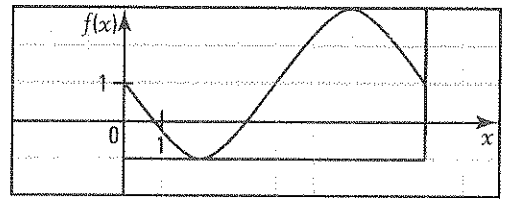
a) $f(x) = \sin \frac{\pi}{3}(x + 4) + 1$ dans $[-4, 2]$.

f ↘ over [−2.5, 0.5]



b) $f(x) = -2 \sin \frac{\pi}{4}x + 1$ over $[0.8]$

f ↘ over [0, 2] ∪ [6, 8]



12. In an experiment, we define a function I which gives the electrical current across a cable, expressed in amperes, as a function of time x, expressed in seconds. The function I is defined by the rule: $I(x) = 4 \sin \frac{\pi}{6}(x + 8) + 4$.

A light bulb lights up when the intensity of the current is equal to 6 amperes.

Determine at what times the light bulb lights up during the first 30 seconds of the experiment.

$4 \sin \frac{\pi}{6}(x + 8) + 4 = 6$

$\sin \frac{\pi}{6}(x + 8) = \frac{1}{2}$

$\frac{\pi}{6}(x + 8) = \frac{\pi}{6}$ or $\frac{\pi}{6}(x + 8) = \frac{5\pi}{6}$

$x = -7$ $\quad$ $x = -3$

The light bulb lights up at the moments 5 s, 9 s, 17 s, 21 s and 29 s.

13. In a company, the number of parts assembled varies according to a sinusoidal function defined by the rule: $f(x) = 200 \sin\left(\frac{\pi}{24}x\right) + 15$ where x represents the number of days elapsed since the beginning of the year.

Over the course of the first 50 days, for how many days was the number of parts assembled greater than or equal to 115?

$200 \sin\left(\frac{\pi}{24}\right)x + 15 = 115$

$\sin \frac{\pi}{24}x = \frac{1}{2}$

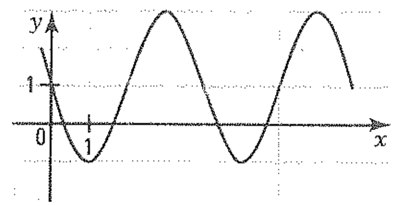
$\frac{\pi}{24}x = \frac{\pi}{6}$ or $\frac{\pi}{24}x = \frac{5\pi}{6}$

$x = 4$ $\quad$ $x = 20$

During 17 days, from the 4th day to the 20th day, inclusively.

ACTIVITY 9 Finding the rule $y = a \sin b(x - h) + k$

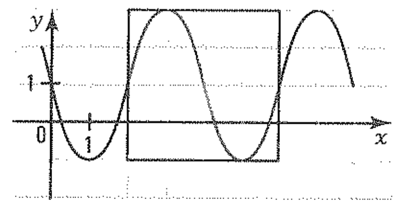
Consider the function f represented on the right.



a) Determine

1. the period of f. ___4___ 2. the amplitude of f. ___2___

b) Determine the rule of the function f when we choose a cycle starting at the point $(h, k) = (2, 1)$.

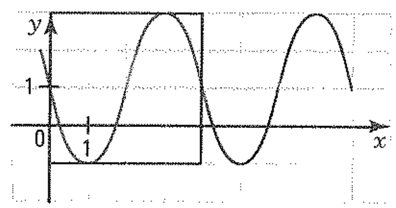


Explain your procedure: ***Since the cycle is increasing from the start, we deduce that $ab > 0$.***

$p = \frac{2\pi}{|b|} = 4 \Rightarrow |b| = \frac{\pi}{2}$; $A = |a| = 2$

Thus, $f(x) = 2 \sin \frac{\pi}{2}(x - 2) + 1$ or $f(x) = -2 \sin \frac{-\pi}{2}(x - 2) + 1$

c) Determine the rule of the function f when we choose a cycle starting at the point $(h, k) = (0, 1)$.



Explain your procedure: ***Since the cycle is decreasing from the start, we deduce that $ab < 0$.***

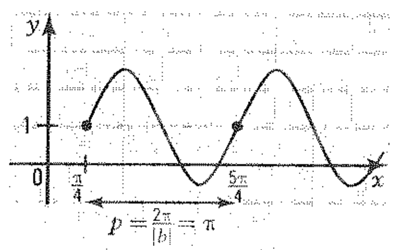
$|b| = \frac{\pi}{2}$; $|a| = 2$

Thus, $f(x) = -2 \sin \frac{\pi}{2}x + 1$ or $f(x) = 2 \sin \frac{-\pi}{2}x + 1$

d) Is it true to say that the writing of the rule varies depending on the chosen cycle but that all of the resulting rules represent the same function? ***Yes***

FINDING THE RULE $y = a \sin b(x - h) + k$

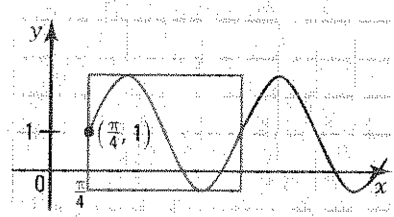
From the graph of the function on the right, we deduce that: $|a| = 1.5$ and $|b| = 2$.



The writing of the function's rule depends on your choice of the starting point (h, k) of one cycle.

Consider two cases:

1st case: $(h, k) = \left(\frac{\pi}{4}, 1\right)$



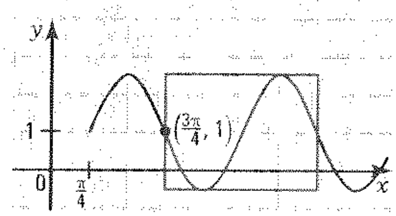
The cycle is increasing from the start $\Rightarrow ab > 0$.

Thus, $a = 1.5$ and $b = 2$ or $a = -1.5$ and $b = -2$.

The rule is written in two ways:

$y = 1.5 \sin 2\left(x - \frac{\pi}{4}\right) + 1$ or $y = -1.5 \sin -2\left(x - \frac{\pi}{4}\right) + 1$.

2nd case: $(h, k) = \left(\frac{3\pi}{4}, 1\right)$



The cycle is decreasing from the start $\Rightarrow ab < 0$.

Thus, $a = 1.5$ and $b = -2$ or $a = -1.5$ and $b = 2$.

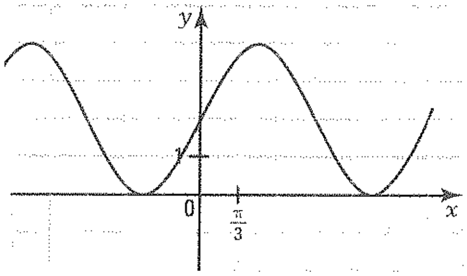
The rule is written in two ways:

$y = 1.5 \sin -2\left(x - \frac{3\pi}{4}\right) + 1$ or $y = -1.5 \sin 2\left(x - \frac{3\pi}{4}\right) + 1$.

Note that the different ways of writing the rule represent the same function.

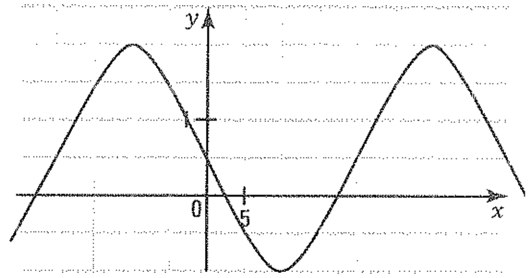
14. Find a rule of the form $y = a \sin b(x - h) + k$ for each of the following functions.

a)



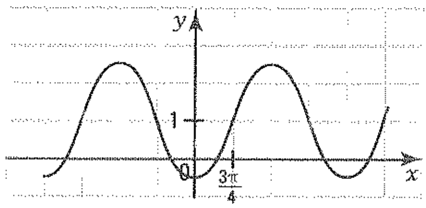
For example, $y = 2 \sin x + 2$

b)



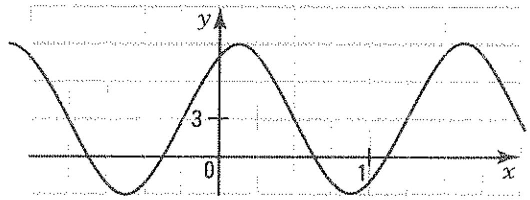
For example, $y = -1.5 \sin \frac{\pi}{20}x + 0.5$

c)



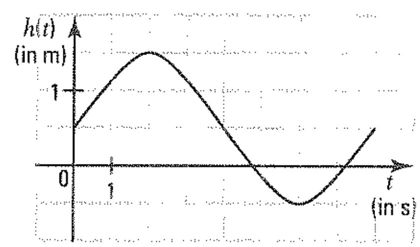
For example, $y = -1.5 \sin \frac{2}{3}\left(x + \frac{3\pi}{4}\right) + 1$

d)



For example, $y = 6 \sin \frac{4\pi}{3}\left(x + \frac{1}{4}\right) + 3$

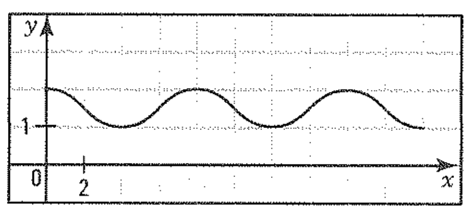
15. The waves of an artificial lake are observed in a laboratory setting as indicated in the figure on the right. The graph represents the movement of one wave where t is time, expressed in seconds, and $h(t)$ is the height of the wave, in metres.



What is the height of the wave after 3 seconds?

$(h, k) = (0, 0.5)$ $h(t) = \sin \frac{\pi}{4}t + 0.5$; $h(3) \approx 1.21$ m

16. Raphael is playing with a yo-yo for 30 seconds. The height of the yo-yo, in metres, varies as a function of time t, in seconds, according to the rule of a sinusoidal function.
Initially, the yo-yo is at its maximum height of 2 m. After 4 seconds, the yo-yo reaches its minimum height of 1 m for the first time.
After how many seconds does Raphael's yo-yo reach a height of 1.8 m for the first time?



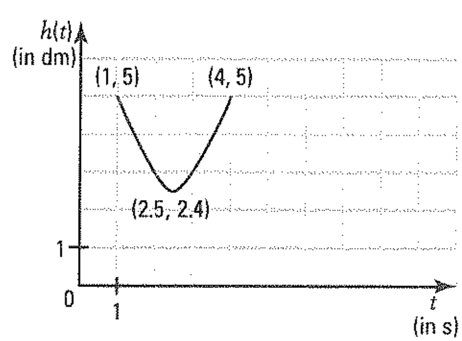
Rule of the function:

for example, $y = -0.5 \sin \frac{\pi}{4}(x - 2) + 1.5$.

$-0.5 \sin \frac{\pi}{4}(x - 2) + 1.5 = 1.8$

$x = 1.18$. After 1.18 seconds, the yo-yo reaches a height of 1.8 m for the first time.

17. The flashing light of an electronic game follows the trajectory of a sinusoidal function as indicated in the graph on the right. If t represents the time (in seconds) and $h(t)$ represents the height (in dm) of the light, determine at what height the light is at the moment $t = 6$ s?



$(h, k) = (1.5)$ ***$y = -2.6 \sin \frac{\pi}{3}(x - 1) + 5$***

$A = 2.6$ $a = -2.6$

$b = \frac{2\pi}{p} = \frac{2\pi}{6} = \frac{\pi}{3}$ ***At the moment $t = 6$ s, the light will be at a height of 7.25 dm.***

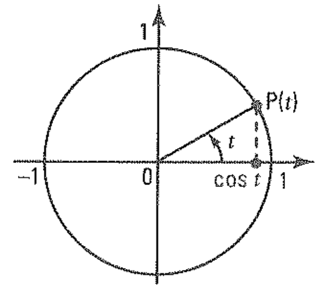
5.6 Cosine function

ACTIVITY 1 Basic cosine function

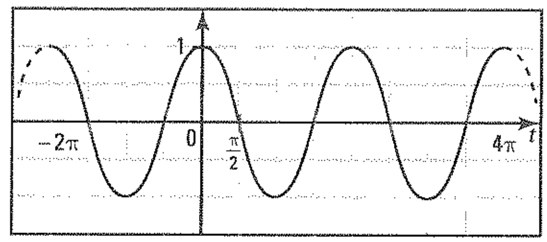
The basic cosine function, denoted cos, has the rule $y = \cos t$.

a) Complete the table of values below when t varies from -2π to 4π.

t	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\cos t$	*1*	*0*	*–1*	*0*	*1*	*0*	*–1*	*0*	*1*	*0*	*–1*	*0*	*1*



b) We have represented the function $y = \cos t$ on the right. Is this function periodic? If yes, what is the period p of the function? ***Yes p = 2π***



c) For this function, determine

1. the domain. ***ℝ***   2. the range. ***[–1, 1]***
3. the maximum. ***1***   4. the minimum. ***–1***

d) What is the amplitude A of the cosine function? ***1***

e) When $t \in [0, 2\pi]$, determine, for the basic cosine function,

1. the zeros. ***$\frac{\pi}{2}$ and $\frac{3\pi}{2}$***
2. the sign. ***$\cos t \geq 0$ if $t \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; $\cos \leq 0$ if $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$***
3. the variation. ***cos ↘ if $t \in [0, \pi]$; cos ↗ if $t \in [\pi, 2\pi]$***

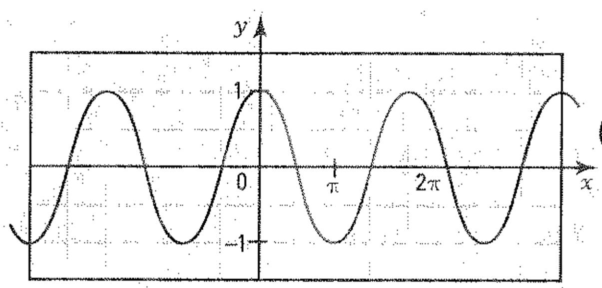
f) Verify, using the graph or the trigonometric circle, the property: $\forall\, t \in \mathbb{R}$: $\cos(-t) = \cos(t)$.

BASIC COSINE FUNCTION

- The cosine function, denoted cos, is defined by

$$\cos : \mathbb{R} \rightarrow \mathbb{R}$$
$$x \mapsto y = \cos x$$

- The cosine function is a periodic function with period 2π. $\cos(x + 2\pi) = \cos x$

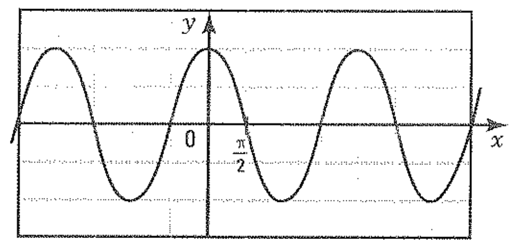


One cycle of the basic cosine function is represented in blue over $[0, 2\pi]$.

- The amplitude of the basic cosine function is: A = 1.

- We have:
  - domain = $\mathbb{R}$, range = $[-1, 1]$.
  - zeros over $[0, 2\pi]$: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.
  - sign over $[0, 2\pi]$: $\cos x \geq 0$ if $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and $\cos x \leq 0$ if $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
  - variation over $[0, 2\pi]$: cos ↘ if $x \in [0, \pi]$; cos ↗ if $x \in [\pi, 2\pi]$.
  - extrema: max = 1; min = −1.
- For any real x, we have: $\cos(-x) = \cos x$. The basic cosine function is therefore considered to be an **even** function.

1. Consider the function $f(x) = \cos x$.



a) Find the zeros of f when

1. $x \in [-2\pi, 4\pi]$. $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ ***and*** $\frac{7\pi}{2}$
2. $x \in \mathbb{R}$. $\left\{\frac{\pi}{2} + \pi n\right\}, n \in \mathbb{Z}$

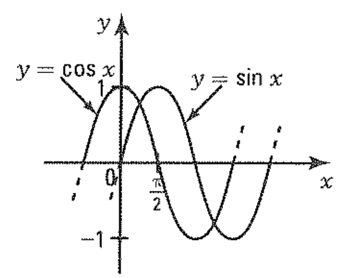
b) Solve the inequality $\cos x \geq 0$ when

1. $x \in \left\{-\frac{5\pi}{2}, \frac{5\pi}{2}\right\}$. $S = \left[\frac{-5\pi}{2}, \frac{-3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$
2. $x \in \mathbb{R}$. $\left[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right]$

c) Find the values of x for which the function f is increasing when

1. $x \in [-2\pi, 4\pi]$. $[-\pi, 0] \cup [\pi, 2\pi] \cup [3\pi, 4\pi]$
2. $x \in \mathbb{R}$. $[\pi + 2\pi n, 2\pi + 2\pi n]$, $n \in \mathbb{Z}$

2. The functions $y = \sin x$ and $y = \cos x$ are represented on the right.



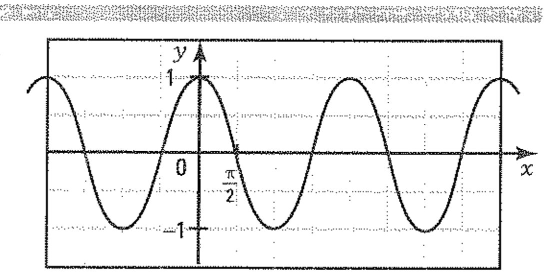
a) Verify that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

b) Complete.

The graph of the function $y = \cos x$ is deduced from the graph of $y = \sin x$ by ***by a phase shift to the left of $\frac{\pi}{2}$ units.***

ACTIVITY 2 Equation $\cos\theta = k$

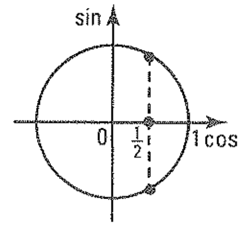
The function $y = \cos x$ is represented on the right when $x \in [-2\pi, 4\pi]$.



a) 1. By referring to the trigonometric circle, solve the equation $\cos\theta = \frac{1}{2}$ when $\theta \in [0, 2\pi]$.

$S = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

2. Explain how to find, from the solutions in 1, the solutions to the equation $\cos\theta = \frac{1}{2}$ when



1) $\theta \in [2\pi, 4\pi]$. ***Add the period 2π to each solution.***
2) $\theta \in [-2\pi, 0]$. ***Subtract the period 2π to each solution.***

3. Verify that the solution set S to the equation $\cos\theta = \frac{1}{2}$ over $\mathbb{R}$ is described by

$S = \left\{\ldots, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots\right\}$ (enumeration).

or by

$S = \left\{\frac{\pi}{3} + 2\pi n\right\} \cup \left\{\frac{5\pi}{3} + 2\pi n\right\}$ where $n \in \mathbb{Z}$ (set-builder notation).

b) By referring to the trigonometric circle and using a calculator, solve the equation $\cos\theta = 0.4$ when

1. $\theta \in [0, 2\pi]$. ***S = {1.16; 5.12}***
2. $\theta \in [4\pi, 6\pi]$. ***S = {13.73; 17.69}***
3. $\theta \in \mathbb{R}$. ***S = {1.16 + 2πn} ∪ {5.12 + 2πn}, n ∈ ℤ***

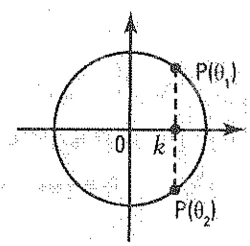
c) Solve the inequality $\cos\theta \geqslant \frac{1}{2}$ when

1. $\theta \in [0, 2\pi]$. $S = \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$
2. $\theta \in [0, 4\pi]$. $S = \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right] \cup \left[2\pi, \frac{7\pi}{3}\right] \cup \left[\frac{11\pi}{3}, 4\pi\right]$
3. $\theta \in \mathbb{R}$. $S = \left[0 + 2\pi n, \frac{\pi}{3} + 2\pi n\right] \cup \left[\frac{5\pi}{3} + 2\pi n, 2\pi + 2\pi n\right], n \in \mathbb{Z}$

EQUATION $\cos\theta = k, \; -1 \leqslant k \leqslant 1$

- When $\theta \in [0, 2\pi[$, the equation $\cos\theta = k$ yields 2 solutions θ_1 and θ_2.

$$\theta_1 = \cos^{-1} k \text{ and } \theta_2 = 2\pi - \theta_1$$



- When $\theta \in \mathbb{R}$, the equation $\cos\theta = k$ yields an infinite number of solutions. The solution set S is defined in set-builder notation by

$$S = \{\theta_1 + 2\pi n\} \cup \{\theta_2 + 2\pi n\} \text{ where } n \in \mathbb{Z}$$

Ex.: The equation $\cos\theta = \frac{\sqrt{3}}{2}$ has the solution set:

– $S = \left\{\frac{\pi}{6}, \frac{11\pi}{6}\right\}$ when $\theta \in [0, 2\pi[$

– $S = \left\{\frac{\pi}{6} + 2\pi n\right\} \cup \left\{\frac{11\pi}{6} + 2\pi n\right\}, n \in \mathbb{Z}$ when $\theta \in \mathbb{R}$.

3. Solve the following equations over

1. $[0, 2\pi]$ 2. $[2\pi, 4\pi]$ 3. $\mathbb{R}$

a) $\cos x = \frac{\sqrt{3}}{2}$ 1. $\left\{\frac{\pi}{6}, \frac{11\pi}{6}\right\}$ 2. $\left\{\frac{13\pi}{6}, \frac{23\pi}{6}\right\}$ 3. $\left\{\frac{\pi}{6} + 2\pi n\right\} \cup \left\{\frac{11\pi}{6} + 2\pi n\right\}, n \in \mathbb{Z}$

b) $\cos x = 1$ 1. ***{0, 2π}*** 2. ***{2π, 4π}*** 3. ***{2πn}, n ∈ ℤ***

c) $\cos x = -\frac{\sqrt{2}}{2}$ 1. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$ 2. $\left\{\frac{11\pi}{4}, \frac{13\pi}{4}\right\}$ 3. $\left\{\frac{3\pi}{4} + 2\pi n\right\} \cup \left\{\frac{5\pi}{4} + 2\pi n\right\}, n \in \mathbb{Z}$

d) $\cos x = 0.6$ 1. $\{0.93; 5.35\}$ 2. $\{7.21; 11.63\}$ 3. $\{0.93 + 2\pi n\} \cup \{5.35 + 2\pi n\}, n \in \mathbb{Z}$

4. Solve the following inequalities over

1. $[0, 2\pi]$ 2. $\mathbb{R}$

a) $\cos x \geqslant \frac{\sqrt{3}}{2}$ 1. $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$ 2. $\left[2\pi n, \frac{\pi}{6} + 2\pi n\right] \cup \left[\frac{11\pi}{6} + 2\pi n, 2\pi + 2\pi n\right]$

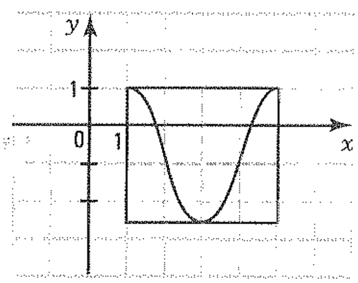
b) $\cos x \leqslant \frac{1}{2}$ 1. $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ 2. $\left[\frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n\right]$

c) $\cos x \leqslant -\frac{1}{2}$ 1. $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ 2. $\left[\frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n\right]$

d) $\cos x \leqslant 1$ 1. $[0, 2\pi]$ 2. $\mathbb{R}$

ACTIVITY 3 Sinusoidal function $y = a \cos b(x - h) + k$

Consider the function $y = 2 \cos \frac{\pi}{2}(x - 1) - 1$.



a) Identify the parameters a, b, h and k.

$a = 2,\ b = \frac{\pi}{2},\ h = 1,\ k = -1$

b) For this function, determine

1. the period. 4 2. the amplitude. 2

c) Draw one cycle of this function using the point $(h, k + A)$ as the starting point.

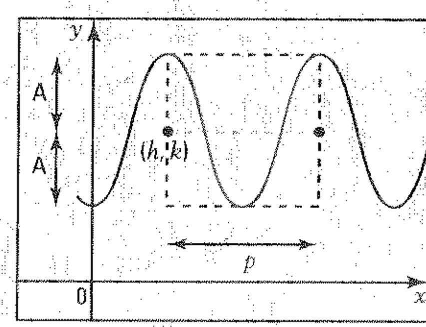
FONCTION SINUSOÏDALE $y = a \cos b(x - h) + k$

- The graph of the function $y = a \cos b(x - h) + k$ is deduced from the graph of the basic function $y = \cos x$ by the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$.
- We have:

  period: $p = \frac{2\pi}{|b|}$; amplitude: $A = |a|$;

  domain $= \mathbb{R}$;

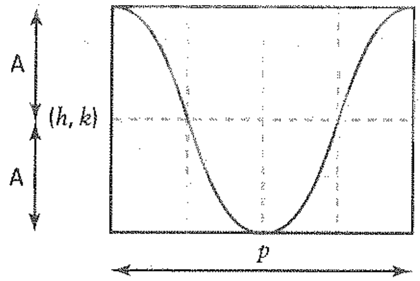
  range $= [k - A, k + A]$

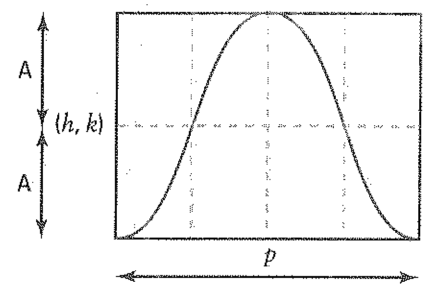


- To draw one cycle of the graph of $y = a \cos b(x - h) + k$,
  1. we identify the point (h, k).
  2. we draw a rectangle with a height equal to 2A, twice the amplitude, and a length equal to the period p.
  3. we draw one cycle inside the rectangle.

There are two situations depending on the sign of a.

$a > 0$ Decreasing cycle from the start. Starting point: $(h, k + A)$.	$a < 0$ Increasing cycle from the start. Starting point: $(h, k - A)$.



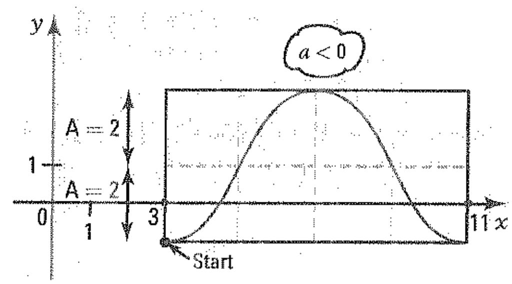


Ex.: Draw one cycle of the function

$y = -2 \cos \frac{-\pi}{4}(x - 3) + 1.$

We have: $a = -2$, $b = \frac{-\pi}{4}$, $h = 3$, $k = 1$

- starting point of the cycle: $(h, k - A) = (3, -1)$.
- amplitude: $A = |a| = 2$.
- period: $p = \frac{2\pi}{|b|} = 8$.

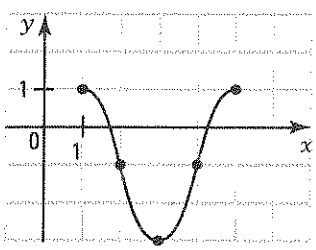


5. Consider the functions $f(x) = a \cos b(x - h) + k$ and $g(x) = a \cos -b(x - h) + k$.
Explain why the functions f and g are equal.

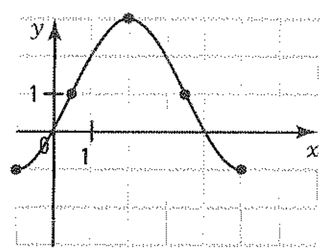
$a \cos -b(x - h) + k = a \cos b(x - h) + k$ since $\cos(-t) = \cos t$.

6. Draw one cycle of each of the following functions.

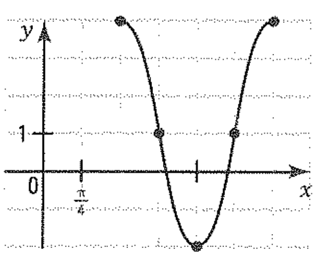
a) $y = 2 \cos \frac{\pi}{2}(x - 1) - 1$



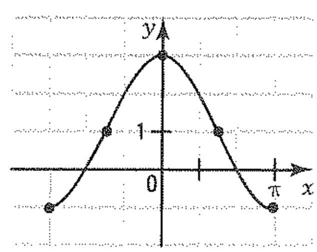
b) $y = -2 \cos \frac{\pi}{3}(x + 1) + 1$



c) $y = 3 \cos 2\left(x - \frac{\pi}{2}\right) + 1$

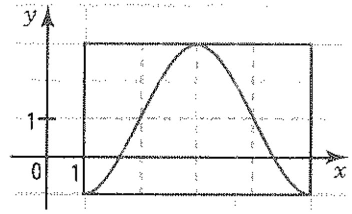


d) $y = -2 \cos -(x + \pi) + 1$



ACTIVITY 4 Finding the zeros of the function $y = a \cos b(x - h) + k$

A portion of the graph of the function $f(x) = -2\cos\frac{\pi}{3}(x-1)+1$ is represented on the right.



a) What is the period p of the function? ***p = 6***

b) 1. How many zeros does the function f have? ***An infinite number.***

2. How many zeros does the function f have when $x \in [1, 7]$?

Two zeros.

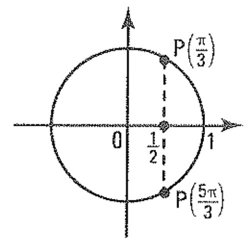
c) Justify the steps for finding the zeros of f when $x \in [1, 7]$.

1. $-2\cos\frac{\pi}{3}(x-1)+1=0$ — ***We set f(x) = 0.***

2. $\cos\frac{\pi}{3}(x-1)=\frac{1}{2}$ — ***We isolate cos b(x − h).***

3. $\frac{\pi}{3}(x-1)=\frac{\pi}{3}$ or $(x-1)=\frac{5\pi}{3}$ — ***We solve the equation $\cos\theta = \frac{1}{2}$ knowing that $\theta = \frac{\pi}{3}(x-1)$.***

4. $x = 2$ or $x = 6$ — ***We deduce the solutions for x.***



d) From the zeros of f obtained over $[1, 7]$, explain how to deduce the zeros of f located on the next cycle, i.e. when $x \in [7, 13]$.

You need to add the period to each zero. The zeros over [7, 13] are 8 and 12.

e) Verify that the set of zeros of f is $\{\ldots, -4, 0, 2, 6, 8, 12, \ldots\}$ or $\{2 + 6n\} \cup \{6 + 6n\}, n \in \mathbb{Z}$.

FINDING THE ZEROS OF THE FUNCTION $f(x) = a \cos b(x - h) + k$

To determine the zeros of $f(x) = a\cos b(x-h)+k$,

1. we establish the period p of the function.

$p = \frac{2\pi}{|b|}$

2. we solve the equation $f(x) = 0$.

- Isolate $\cos b(x - h)$.
- Find the angles $b(x - h)$ over the interval $[0, 2\pi]$ that verify the equation.
- Isolate x.
- Determine the solution set, taking the period into consideration.

Ex.: $f(x) = 2\cos\frac{\pi}{2}(x-1)-1$

$p = 4$

$2\cos\frac{\pi}{2}(x-1)-1=0$

$\cos\frac{\pi}{2}(x-1)=\frac{1}{2}$

$\frac{\pi}{2}(x-1)=\frac{\pi}{3}$ or $\frac{\pi}{2}(x-1)=\frac{5\pi}{3}$

$x=\frac{5}{3}$ or $x=\frac{13}{3}$

$S=\left\{\frac{5}{3}+4n\right\}\cup\left\{\frac{13}{3}+4n\right\}$

7. The following functions have the rule $f(x) = a\cos b(x - h) + k$.
Find the zeros of each function over
1. the interval $[h, h + p]$ where p is the period of the function.
2. the set of all real numbers.

a) $f(x) = 2\cos\frac{\pi}{6}(x - 2) + 1$

$\cos\frac{\pi}{6}(x - 2) = -\frac{1}{2}$

$\frac{\pi}{6}(x - 2) = \frac{2\pi}{3}$ or $\frac{\pi}{6}(x - 2) = \frac{4\pi}{3}$

$x = 6$ or $x = 10$

1. $S = \{6, 10\}$
2. $S = \{6 + 12n\} \cup \{10 + 12n\}$

b) $f(x) = -2\cos\frac{\pi}{3}(x + 1) + \frac{1}{2}$

$\cos\frac{\pi}{3}(x + 1) = \frac{1}{4}$

$\frac{\pi}{3}(x + 1) = 1.32$ or $\frac{\pi}{3}(x + 1) = 4.97$

$x = 0.26$ or $x = 3.75$

1. $S = \{0.26; 3.75\}$
2. $S = \{0.26 + 6n\} \cup \{3.75 + 6n\}$

c) $f(x) = \cos 2(x - \pi) + 1$

$\cos 2(x - \pi) = -1$

$2(x - \pi) = \pi$

$x = \frac{3\pi}{2}$

1. $S = \left\{\frac{3\pi}{2}\right\}$
2. $S = \left\{\frac{3\pi}{2} + \pi n\right\}$

d) $f(x) = 2\cos\left(x + \frac{\pi}{2}\right) - \sqrt{3}$

$\cos\left(x + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$

$x + \frac{\pi}{2} = \frac{\pi}{6}$ or $x + \frac{\pi}{2} = \frac{11\pi}{6}$

$x = -\frac{\pi}{3}$ or $x = \frac{4\pi}{3}$

1. $S = \left\{\frac{-\pi}{3}, \frac{4\pi}{3}\right\}$
2. $S = \left\{\frac{-\pi}{3} + 2\pi n\right\} \cup \left\{\frac{4\pi}{3} + 2\pi n\right\}$

e) $f(x) = -3\cos\frac{\pi}{4}x + 6$

$\cos\frac{\pi}{4}x = 2$

1. $S = \varnothing$
2. $S = \varnothing$

f) $f(x) = -2\cos\frac{\pi}{8}(x + 2) - \sqrt{2}$

$\frac{\pi}{8}(x + 2) = \frac{3\pi}{4}$ or $\frac{\pi}{8}($

$\frac{\pi}{8}(x + 2) = \frac{3\pi}{4}$ or $\frac{\pi}{8}(x + 2) = \frac{5\pi}{4}$

$x = 4$ $x = 8$

1. $S = \{4, 8\}$
2. $S = \{4 + 16n\} \cup \{8 + 16n\}$

8. Determine the zeros of the function $f(x) = -2\cos\frac{\pi}{12}(x + 5) - 1$ over the interval [66, 126].

$\cos\frac{\pi}{12}(x + 5) = -\frac{1}{2}$

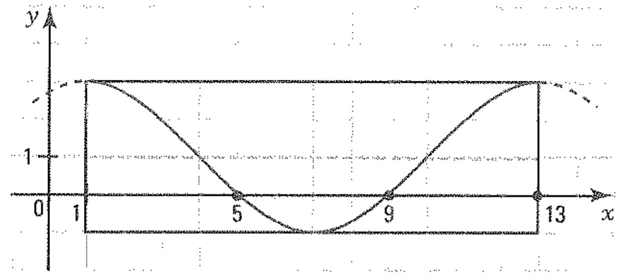
$\frac{\pi}{12}(x + 5) = \frac{2\pi}{3}$ or $\frac{\pi}{12}(x + 5) = \frac{4\pi}{3}$

$x = 3$ or $x = 11$

The zeros over the interval [66, 126] are 75, 83, 99, 107, 123.

ACTIVITY 5 Study of the function $f(x) = a \cos b(x - h) + k$

One cycle of the function $f(x) = 2\cos\frac{\pi}{6}(x-1)+1$ is represented below.



a) Determine

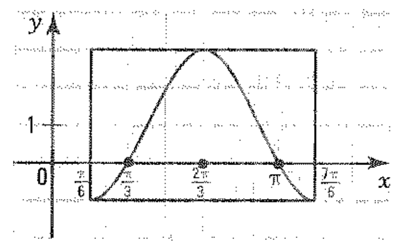
1. the period. *12*   2. the amplitude. *2*

b) Determine

1. dom f. $\mathbb{R}$ 2. ran f. *[-1, 3]*
3. the zeros of f over [1, 13]. *5 and 9*
4. the zeros of f over $\mathbb{R}$. *{5 + 12n} ∪ {9 + 12n}*
5. the sign of f over [1, 13]. *f(x) ≥ 0 over [1, 5] ∪ [9, 13] and f(x) ≤ 0 over [5, 9]*
6. the sign of f over $\mathbb{R}$. *f(x) ≥ 0 over [1 + 12n, 5 + 12n] ∪ [9 + 12n, 13 + 12n]*
   f(x) ≤ 0 over [5 + 12n, 9 + 12n]
7. the variation of f over [1, 13]. *f ↘ 0 over [1, 7] and f ↗ 0 over [7, 13]*
8. the variation of f over $\mathbb{R}$. *f ↘ 0 over [1 + 12n, 7 + 12n] and f ↗ 0 over [7 + 12n, 13 + 12n]*
9. the maximum and minimum of f. *max f = 3; min f = -1*

STUDY OF THE FUNCTION $f(x) = a \cos b(x - h) + k$

Given $f(x) = -2\cos 2\left(x - \frac{\pi}{6}\right) + 1$. We have:



- dom $f = \mathbb{R}$; ran $f = [-1, 3]$
- period $p = \pi$; amplitude $A = 2$
- zeros of f: $\left\{\frac{\pi}{3} + \pi n\right\} \cup \{\pi + \pi n\}$
- sign of f.

  $f(x) \leq 0$ over $\left[\frac{\pi}{6} + \pi n, \frac{\pi}{3} + \pi n\right] \cup \left[\pi + \pi n, \frac{7\pi}{6} + \pi n\right]$

  $f(x) \geq 0$ over $\left[\frac{\pi}{3} + \pi n, \pi + \pi n\right]$
- variation of f.

  $f \nearrow$ over $\left[\frac{\pi}{6} + \pi n, \frac{2\pi}{3} + \pi n\right]$; $f \searrow$ over $\left[\frac{2\pi}{3} + \pi n, \frac{7\pi}{6} + \pi n\right]$
- min $f = -1$; max $f = 3$.

9. For each of the following functions, determine

1. the period,   2. the amplitude,   3. the range of the function.

a) $f(x) = 2\cos\frac{3\pi}{4}(x-1)+5$

1. *$p = \frac{8}{3}$*
2. *A = 2*
3. *ran f = [3, 7]*

b) $f(x) = 5\cos 4(x+2) - 12$

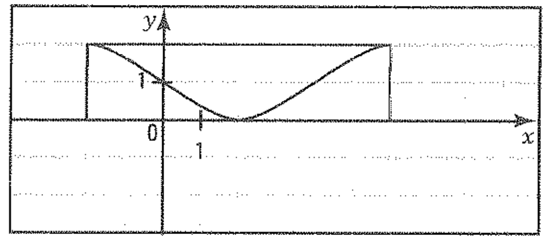
1. *$p = \frac{\pi}{2}$*
2. *A = 5*
3. *ran f = [-17, -7]*

10. Determine the initial value of the following functions.

a) $f(x) = 2\cos\frac{\pi}{3}(x+1)+1$ ***2*** b) $f(x) = 5\cos 2\left(x-\frac{\pi}{2}\right)+1$ ***-4***

11. Determine, over $\mathbb{R}$, the interval over which the function $f(x) = \cos\frac{\pi}{4}(x+2)+1$ is increasing.

$f \nearrow$ over $[2 + 8n,\ 6 + 8n]$



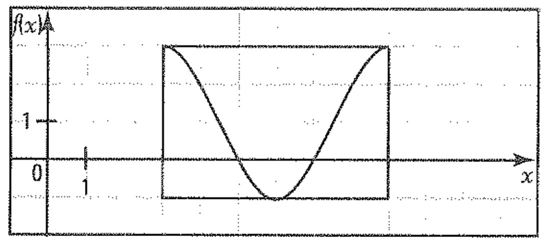
12. Determine, over $\mathbb{R}$, the interval over which the function $f(x) = 2\cos\frac{\pi}{3}(x-3)+1$ is negative.

Zeros: $\cos\frac{\pi}{3}(x-3) = -\frac{1}{2}$

$\frac{\pi}{3}(x-3) = \frac{2\pi}{3}$ or $\frac{\pi}{3}(x-3) = \frac{4\pi}{3}$

$x = 5$ or $x = 7$

$f(x) \leq 0$ over $[5 + 6n,\ 7 + 6n]$



13. On a boat, a sailor observes the movement of the waves while on a sailing expedition. The height of a wave (in m) can be expressed as a function of the time (in s) since the start of the observation by the rule:

$$h(t) = 2\cos\frac{\pi}{6}(t-4)+1$$

The sailor observes a wave for 30 seconds. Determine at which moments the wave will be at a height of 2 metres.

$2\cos\frac{\pi}{6}(t-4)+1 = 2$

$\cos\frac{\pi}{6}(t-4) = \frac{1}{2}$

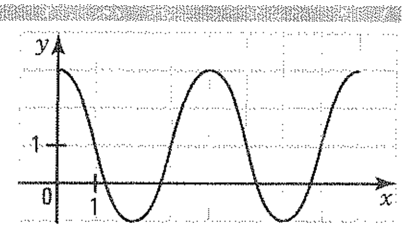
$\frac{\pi}{6}(t-4) = \frac{\pi}{3}$ or $\frac{\pi}{6}(t-4) = \frac{5\pi}{3}$

$t = 6$ or $t = 14$

The wave will be at a height of 2 m at the moments $t = 2$ s, $t = 6$ s, $t = 14$ s, $t = 18$ s and $t = 26$ s.

ACTIVITY 6 Finding the rule $y = a\cos b(x-h)+k$

Consider the function f represented on the right.



a) Determine

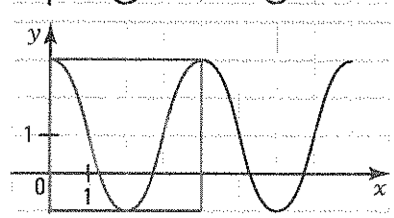
1. the period of f. ***4*** 2. the amplitude of f. ***2***

b) Determine the rule of the function f when we choose a cycle starting at the point $(h, k + A) = (0, 3)$.

Explain your procedure: ***Since the cycle is decreasing from the start, we deduce that $a > 0$. Thus, $a = 2$ and $k = 1$***

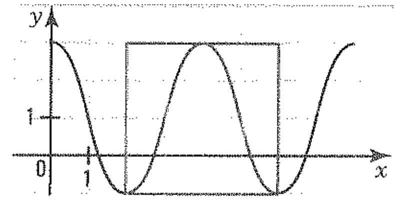
$p = 4 \Rightarrow |b| = \frac{\pi}{2}$.

Thus, $f(x) = 2\cos\frac{\pi}{2}x + 1$ ***or*** $f(x) = 2\cos\frac{-\pi}{2}x + 1$



c) Determine the rule of the function f when we choose a cycle starting at the point $(h, k - A) = (2, -1)$.

$y = -2\cos\frac{\pi}{2}(x - 2) + 1$



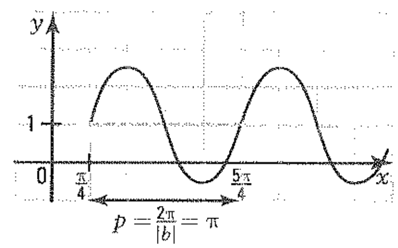
d) Write the rule of the function in the form $y = a \sin b(x - h) + k$.

For example, $y = -2\sin\frac{\pi}{2}(x - 1) + 1$

FINDING THE RULE $y = a\cos b(x - h) + k$

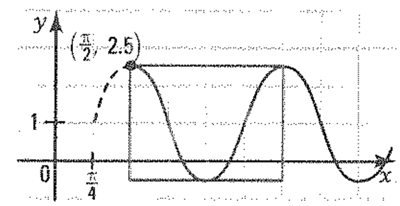
From the graph of the function on the right, we deduce that: $|a| = 1.5$, $|b| = 2$ and $k = 1$.

The writing of the function's rule depends on your choice of the starting point of one cycle.



Consider two cases:

1st case: starting point: $(h, k + A) = \left(\frac{\pi}{2}, 2.5\right)$



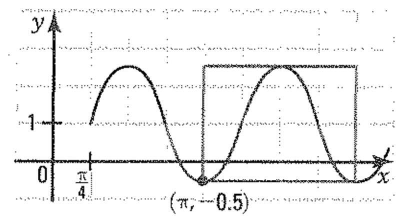
The cycle is decreasing from the start $\Rightarrow a > 0$.

Thus, $a = 1.5$ and $b = 2$ or $a = 1.5$ and $b = -2$.

The rule is written in two ways:

$y = 1.5\cos 2\left(x - \frac{\pi}{2}\right) + 1$ or $y = 1.5\cos -2\left(x - \frac{\pi}{2}\right) + 1$.

2nd case: starting point: $(h, k - A) = (\pi, -0.5)$.



The cycle is increasing from the start $\Rightarrow a < 0$.

Thus, $a = -1.5$ and $b = 2$ or $a = -1.5$ and $b = -2$.

The rule is written in two ways:

$y = -1.5\cos 2(x - \pi) + 1$ or $y = -1.5\cos -2(x - \pi) + 1$.

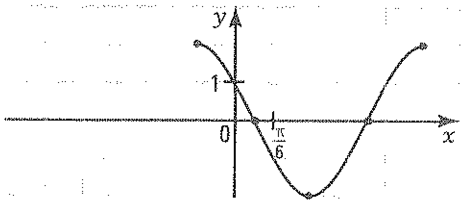
Note that this function can be described by a rule written in the form $y = a\sin b(x - h) + k$.

For example, $y = 1.5\sin 2\left(x - \frac{\pi}{4}\right) + 1$.

Thus, any sinusoidal function can be described by a rule written in the form $y = a\sin b(x - h) + k$ or $y = a\cos b(x - h) + k$.

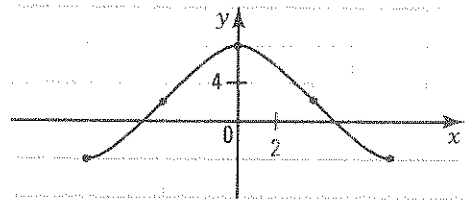
14. Find a rule of the form $y = a \cos b(x - h) + k$ for each of the following functions.

a)



For example, $y = 2 \cos 2\left(x + \frac{\pi}{6}\right)$

b)



For example, $y = -6 \cos \frac{\pi}{8}(x + 8) + 2$

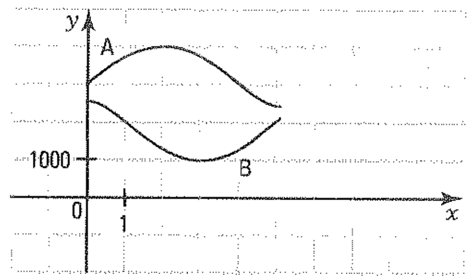
15. The populations of two neighboring villages A and B vary according to the model of a sinusoidal function which gives the population P of the village as a function of time t, in years, since the year 2000.

In the year 2000, the two villages had 3000 and 2500 inhabitants respectively.

The graph on the right shows the progression of the population of each village.

Village A reaches its maximum population of 4125 after 2 years and village B reaches its minimal population of 1000 after 3 years.

What will be the difference in population between these two villages in the year 2005?



Rule corresponding to village A.	*Rule corresponding to village B.*
$(h, k) = (0, 3000)$	*$(h, k + A) = (0, 2500)$*
$A = 1125$	*$A = 750$*
$p = 8 \Rightarrow b = \frac{\pi}{4}$	*$p = 6 \Rightarrow b = \frac{\pi}{3}$*
$y = 1125 \sin \frac{\pi}{4}x + 3000$	*$y = 750 \cos \frac{\pi}{3}x + 1750$*
In 2005, $y = 2205$ inhabitants	*In 2005, $y = 2125$ inhabitants*

The difference in their populations will be 80 inhabitants.

5.7 Tangent function

ACTIVITY 1 Basic tangent function

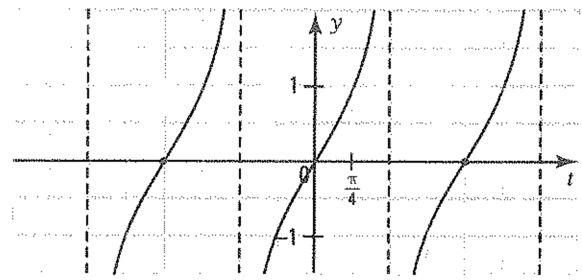
The **basic tangent** function, denoted **tan**, has the rule $y = \tan t$.

a) Complete the table of values below when t varies from $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$.

t	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
tan t		*−1*	*0*	*1*		*−1*	*0*	*1*		*−1*	*0*	*1*	

b) Explain why $\tan t$ does not exist when $t = \frac{\pi}{2}$.

$\tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \frac{1}{0}$. Division by zero is not defined.



c) We have represented the function $y = \tan t$ on the right. Is this function periodic? If yes, what is the period p of the function? *$p = \pi$*

d) The graph of the tangent function has vertical asymptotes. What are the equations of the tangent function's asymptotes when $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$? *$t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$*

e) When $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, determine, for the basic tangent function,

1. the domain. *$\left]\frac{\pi}{2}, \frac{\pi}{2}\right[$*
2. the range. *$\mathbb{R}$*
3. the zero. *0*
4. the sign. *$\tan t \leqslant 0$ if $-\frac{\pi}{2} < t \leqslant 0$ and $\tan t \geqslant 0$ if $0 \leqslant t < \frac{\pi}{2}$.*
5. the variation. *$\tan \nearrow$ if $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$*

f) Does the basic tangent function have any extrema? *No*

BASIC TANGENT FUNCTION

- The **tangent** function, denoted tan, is defined by

$$\tan : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = \tan x$$

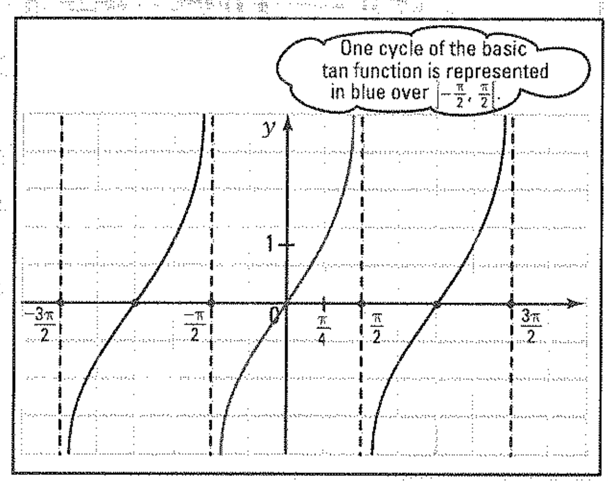
- The basic tangent function is a **periodic** function with period π.

$$\boxed{\tan(x + \pi) = \tan x}$$

- The tangent function has an infinite number of vertical asymptotes with equations:

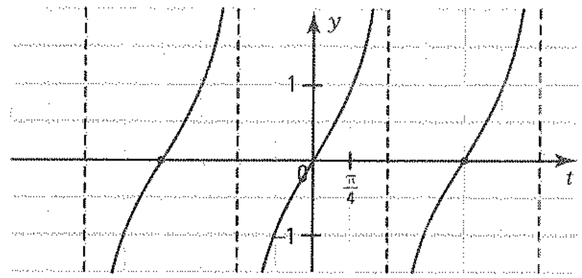
$x = \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z})$.

Note that the distance separating two consecutive asymptotes is the period π.



- We have:
  - domain $= \mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi n\right\}$; range $= \mathbb{R}$.
  - zero over $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$: 0.
  - sign over $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$: $\tan x \leqslant 0$ if $-\frac{\pi}{2} < x \leqslant 0$ and $\tan x \geqslant 0$ if $0 \leqslant x < \frac{\pi}{2}$.
  - variation: tan ↗ over $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
  - extrema: none
- For any real x, we have: $\tan(-x) = -\tan x$. The basic tangent function is therefore considered to be an **odd** function.

1. Consider the function $y = \tan x$.



a) Find the equations of the asymptotes when

1. $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. ***$x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$***
2. $x \in \mathbb{R}$. ***$x = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$***

b) Find the zeros of f when

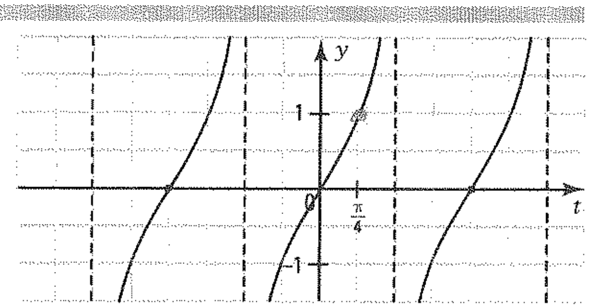
1. $x \in \left]-\frac{3\pi}{2}, \frac{3\pi}{2}\right[$. ***$-\pi,\ 0, \pi$***
2. $x \in \mathbb{R}$. ***$\{n\pi\},\ n \in \mathbb{Z}$***

c) Solve the inequality $\tan x \geqslant 0$ when

1. $x \in \left]-\frac{3\pi}{2}, \frac{3\pi}{2}\right[$. ***$\left[-\pi, \frac{\pi}{2}\right[\cup \left[0, \frac{\pi}{2}\right[\cup \left[\pi, \frac{3\pi}{2}\right[$***
2. $x \in \mathbb{R}$. ***$\left[\pi n, \frac{\pi}{2} + \pi n\right[$***

ACTIVITY 2 Equation $\tan\theta = k$

The function $y = \tan x$ is represented on the right when $x \in \left]-\frac{3\pi}{2}, \frac{3\pi}{2}\right[$.

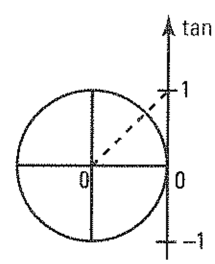


a) 1. By referring to the trigonometric circle, solve the equation $\tan\theta = 1$ when $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. ***$S = \left\{\frac{\pi}{4}\right\}$***

2. Explain how to find, from the solutions in 1, the solution to $\tan\theta = 1$ when

1) $\theta \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$. ***Add the period π to the solution.***

2) $\theta \in \left]-\frac{3\pi}{2}, -\frac{\pi}{2}\right[$. ***Subtract the period π from the solution.***



3. Verify that the solution set S to the equation $\tan\theta = 1$ over $\mathbb{R}$ is described by

$S = \left\{\dots, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \dots\right\}$ (enumeration).

or $S = \left\{\frac{\pi}{4} + \pi n\right\}$ where $n \in \mathbb{Z}$ (set-builder notation).

b) Using a calculator, solve the equation $\tan\theta = 2$ when

1. $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. ***1.11*** 2. $\theta \in \left]\frac{3\pi}{2}, \frac{5\pi}{2}\right[$. ***7.39*** 3. $\theta \in \mathbb{R}$. $\boldsymbol{1.11 + \pi n}$

c) Solve the inequality $\tan\theta \geqslant 2$ when

1. $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. $\left[1.11; \frac{\pi}{2}\right[$
2. $\theta \in \left]\frac{3\pi}{2}, \frac{5\pi}{2}\right[$. $\left[7.39; \frac{5\pi}{2}\right[$
3. $\theta \in \mathbb{R}$. $\left[1.11 + \pi n; \frac{\pi}{2} + \pi n\right[$

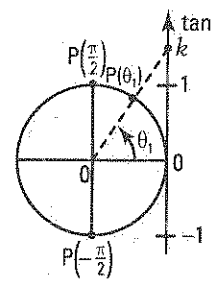
EQUATION $\tan\theta = k, k \in \mathbb{R}$

- When $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, the equation $\tan\theta = k$ yields one **unique solution**

$$\theta_1 = \tan^{-1} k$$

- When $\theta \in \mathbb{R}$, the equation $\tan\theta = k$ yields an infinite number of solutions. The solution set S is defined in set-builder notation by

$$S = \{\theta_1 + \pi n\} \text{ where } n \in \mathbb{Z}$$



Ex.: The equation $\tan\theta = \sqrt{3}$ has the solution set:

– $S = \left\{\frac{\pi}{3}\right\}$ when $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

– $S = \left\{\frac{\pi}{3} + \pi n\right\}$ when $\theta \in \mathbb{R}$.

2. Solve the following equations over

1. $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ 2. $\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ 3. $\mathbb{R}$

a) $\tan x = \frac{\sqrt{3}}{3}$ 1. $S = \left\{\frac{\pi}{6}\right\}$ 2. $S = \left\{\frac{7\pi}{6}\right\}$ 3. $S = \left\{\frac{\pi}{6} + \pi n\right\}$

b) $\tan x = -1$ 1. $S = \left\{-\frac{\pi}{4}\right\}$ 2. $S = \left\{\frac{3\pi}{4}\right\}$ 3. $S = \left\{-\frac{\pi}{4} + \pi n\right\}$

c) $\tan x = -\sqrt{3}$ 1. $S = \left\{-\frac{\pi}{3}\right\}$ 2. $S = \left\{\frac{2\pi}{3}\right\}$ 3. $S = \left\{-\frac{\pi}{3} + \pi n\right\}$

d) $\tan x = 5$ 1. $S = \{1.37\}$ 2. $S = \{4.51\}$ 3. $S = \{1.37 + \pi n\}$

3. Solve the following inequalities over

1. $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ 2. $\mathbb{R}$

a) $\tan x \geqslant \sqrt{3}$ 1. $S = \left[\frac{\pi}{3}, \frac{\pi}{2}\right[$ 2. $S = \left[\frac{\pi}{3} + \pi n, \frac{\pi}{2} + \pi n\right[$

b) $\tan x \leqslant \frac{\sqrt{3}}{3}$ 1. $S = \left]-\frac{\pi}{2}, \frac{\pi}{6}\right]$ 2. $S = \left]-\frac{\pi}{2} + \pi n, \frac{\pi}{6} + \pi n\right]$

c) $\tan x \leqslant -1$ 1. $S = \left]-\frac{\pi}{2}, -\frac{\pi}{4}\right]$ 2. $S = \left]-\frac{\pi}{2} + \pi n, -\frac{\pi}{4} + \pi n\right]$

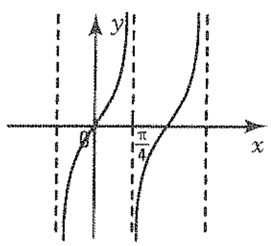
d) $\tan x \geqslant -\sqrt{3}$ 1. $S = \left[-\frac{\pi}{3}, \frac{\pi}{2}\right[$ 2. $S = \left[-\frac{\pi}{3} + \pi n, \frac{\pi}{2} + \pi n\right[$

ACTIVITY 3 Function $y = \tan bx$

a) For each of the following functions,

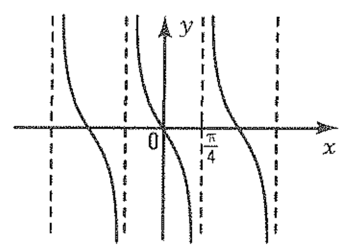
1. determine the value of parameter b.
2. find the period p.

a) $y = \tan 2x$



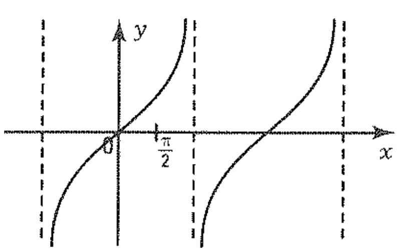
1. $b = 2$ 2. $p = \frac{\pi}{2}$

b) $y = \tan(-2x)$



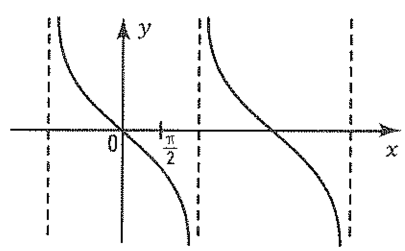
1. $b = -2$ 2. $p = \frac{\pi}{2}$

c) $y = \tan \frac{1}{2}x$



1. $b = \frac{1}{2}$ 2. $p = 2\pi$

d) $y = \tan\left(-\frac{1}{2}x\right)$



1. $b = \frac{-1}{2}$ 2. $p = 2\pi$

b) What is the period p of the function $y = \tan bx$? $p = \frac{\pi}{|b|}$

ACTIVITY 4 Tangent function $y = a \tan b(x - h) + k$

a) Consider the function $y = 2 \tan \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 1$.

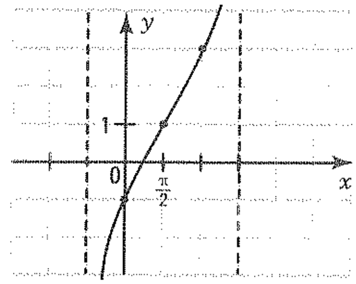
1. Identify the parameters a, b, h and k. $a = 2,\ b = \frac{1}{2},\ h = \frac{\pi}{2},\ k = 1$
2. What is the period p of the function? $p = \frac{\pi}{|b|} = 2\pi$

b) To determine the equation of an asymptote, solve the equation $b(x - h) = \frac{\pi}{2}$.

1. Find the equation of an asymptote. $\frac{1}{2}\left(x - \frac{\pi}{2}\right) = \frac{\pi}{2};\quad x = \frac{3\pi}{2}$
2. What distance separates two consecutive asymptotes? *The period $p = 2\pi$.*

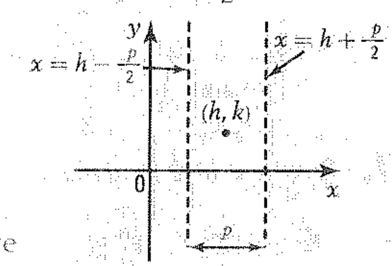
c) In the Cartesian plane on the right, the point (h, k) has been located and 2 consecutive asymptotes are drawn.
Complete the following table of values and draw one cycle of the function.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
y		-1	1	3	



TANGENT FUNCTION $y = a \tan b(x - h) + k$

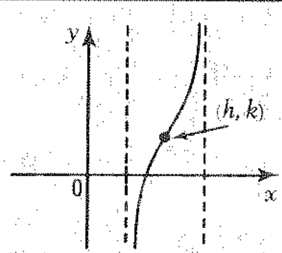
- The graph of the function $y = a \tan b(x - h) + k$ is deduced from the graph of the basic function $y = \tan x$ by the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$.
- The function is periodic with period: $p = \frac{\pi}{|b|}$.
- The function has an infinite number of vertical asymptotes.
  - To determine the equation of an asymptote, we solve the equation $b(x - h) = \frac{\pi}{2}$.
  - Equation of an asymptote: $x = h - \frac{p}{2}$ or $x = h + \frac{p}{2}$.
  - The distance separating two consecutive asymptotes is equal to the period p of the function.

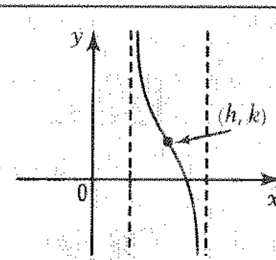


- To draw one cycle of the function $y = a \tan b(x - h) + k$,
  1. we locate the point (h, k).
  2. we draw, equal distance from the point (h, k), two consecutive asymptotes.

  We have two possible situations depending on the signs of a and b.

$ab > 0$ Increasing function	$ab < 0$ Decreasing function





4. For each of the following functions, determine
1. the period. 2. the equations of 2 consecutive asymptotes. 3. the domain.

a) $f(x) = 2\tan\frac{\pi}{4}(x-1)+1$

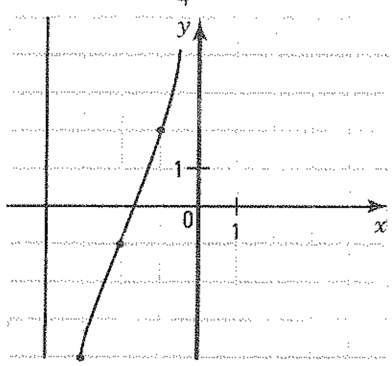
1. $p = 4$
2. $x = -1$ and $x = 3$
3. $\mathbb{R}\setminus\{3 + 4n\}$

b) $f(x) = -3\tan\frac{2\pi}{3}(x+2)-5$

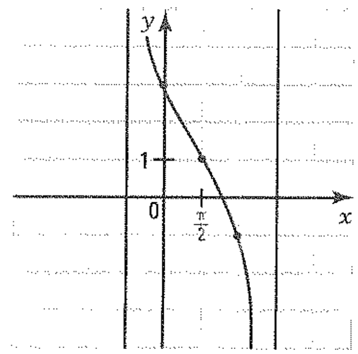
1. $p = \frac{3}{2}$
2. $x = \frac{-11}{4}$ and $x = -\frac{5}{4}$
3. $\mathbb{R}\setminus\left\{\frac{-5}{4} + \frac{3}{2}n\right\}$

5. Draw one cycle of each of the following functions.

a) $f(x) = 3\tan\frac{\pi}{4}(x+2)-1$



b) $f(x) = -2\tan\frac{1}{2}\left(x-\frac{\pi}{2}\right)+1$



6. Determine the zeros of the following functions over $\mathbb{R}$.

a) $f(x) = -2\tan\frac{\pi}{6}(x-1)+2$

$\tan\frac{\pi}{6}(x-1) = 1$

$\frac{\pi}{6}(x-1) = \frac{\pi}{4}$

$x = \frac{5}{2}$

zeros: $\frac{5}{2} + 6n$

b) $f(x) = 3\tan\frac{\pi}{3}(x+2)+\sqrt{3}$

$\tan\frac{\pi}{3}(x+2) = -\frac{\sqrt{3}}{3}$

$\frac{\pi}{3}(x+2) = -\frac{\pi}{6}$

$x = -\frac{5}{2}$

zeros: $-\frac{5}{2} + 3n$

7. Determine the initial value of the following functions.

a) $f(x) = 3\tan\frac{\pi}{2}\left(x+\frac{1}{2}\right)-1$

2

b) $f(x) = 3\tan\frac{1}{2}\left(x-\frac{\pi}{2}\right)+2$

-1

8. Determine the interval over which the function $f(x) = 4\tan\frac{\pi}{4}(x-1)+4$ is positive.

zero: $\tan\frac{\pi}{4}(x-1) = -1$

$\frac{\pi}{4}(x-1) = -\frac{\pi}{4}$

$x = 0$

$f(x) \geq 0$ over the interval $[0 + 4n, 3 + 4n[$.

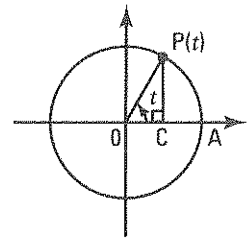
9. Determine the variation of the function $f(x) = -2\tan\frac{\pi}{6}(x+4)-1$.

The function is decreasing over $\mathbb{R}\setminus\{-1 + 6n\}$ since $ab < 0$.

5.8 Trigonometric identities

ACTIVITY 1 Trigonometric identities

Consider the trigonometric point P(t) and the right triangle 0PC.



a) Prove the identity $\sin^2 t + \cos^2 t = 1$.

Since $m\overline{0P} = 1$ (radius of the unit circle),

We have: $\sin t = m\overline{CP}$ and $\cos t = m\overline{0C}$.

Since $m\overline{CP}^2 + m\overline{0C}^2 = m\overline{0P}^2$ (Pythagorean theorem), we deduce that

$\sin^2 t + \cos^2 t = 1$.

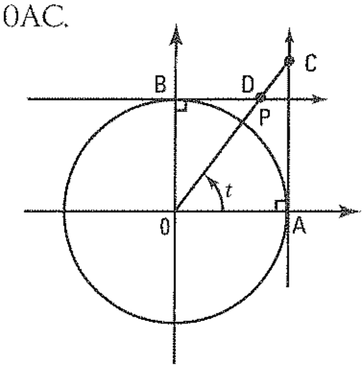
b) Justify the steps proving the identity $\tan^2 t + 1 = \sec^2 t$.

Steps	Justifications
1. $\sin^2 t + \cos^2 t = 1$	***See a)***
2. $\dfrac{\sin^2 t + \cos^2 t}{\cos^2 t} = \dfrac{1}{\cos^2 t}$	***Divide each side by $\cos^2 t$.***
3. $\dfrac{\sin^2 t}{\cos^2 t} + \dfrac{\cos^2 t}{\cos^2 t} = \dfrac{1}{\cos^2 t}$	***Apply the property: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.***
4. $\tan^2 t + 1 = \sec^2 t$	***$\tan t = \frac{\sin t}{\cos t}$ and $\sec t = \frac{1}{\cos t}$.***

c) Prove the identity: $1 + \cot^2 t = \csc^2 t$.

$\sin^2 t + \cos^2 t = 1 \Rightarrow \dfrac{\sin^2 t + \cos^2 t}{\sin^2 t} = \dfrac{1}{\sin^2 t} \Rightarrow 1 + \cot^2 t = \csc^2 t$.

d) Consider the trigonometric circle on the right and the right triangle 0AC.
Justify the steps showing that $m\overline{0C} = \sec t$.



Steps	Justifications
1. $m\overline{0C}^2 = m\overline{0A}^2 + m\overline{AC}^2$	***Pythagorean theorem***
2. $m\overline{0C}^2 = 1 + \tan^2 t$	***$m\overline{0A} = 1$ and $m\overline{AC} = \tan t$***
3. $m\overline{0C}^2 = \sec^2 t$	***$1 + \tan^2 t = \sec^2 t$ (See b)***
4. $m\overline{0C} = \sec t$	***Consequence***

e) Refer to the trigonometric circle and right triangle 0BD above to show that $m\overline{0D} = \csc t$.

$m\overline{0D}^2 = m\overline{0B}^2 + m\overline{BD}^2$ (Pythagorean theorem).

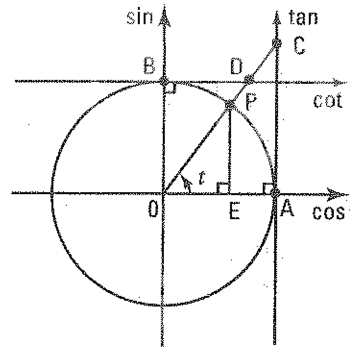
$m\overline{0D}^2 = 1^2 + \cot^2 t$ ($m\overline{0B} = 1$ and $m\overline{BD} = \cot t$)

$m\overline{0D}^2 = \csc^2 t$ (See c)

$m\overline{0D} = \csc t$

BASIC TRIGONOMETRIC IDENTITIES

- If P(t) is a trigonometric point, then
  $m\overline{0E} = \cos t$, $m\overline{PE} = \sin t$.
  $m\overline{AC} = \tan t$, $m\overline{BD} = \cot t$.
  $m\overline{0C} = \sec t$, $m\overline{0D} = \csc t$.



- 1st basic identity:

$$\sin^2 t + \cos^2 t = 1$$

- Other basic identities:

$$1 + \tan^2 t = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

1. Verify the three basic identities when $t = \frac{\pi}{6}$.

$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6} = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$1 + \tan^2 \frac{\pi}{6} = 1 + \left(\frac{\sqrt{3}}{3}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}; \sec^2 \frac{\pi}{6} = \frac{1}{\cos^2 \frac{\pi}{6}} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$1 + \cot^2 \frac{\pi}{6} = 1 + \left(\frac{3}{\sqrt{3}}\right)^2 = 1 + 3 = 4; \csc^2 \frac{\pi}{6} = \frac{1}{\sin^2 \frac{\pi}{6}} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4.$$

2. Using an angle measure t of your choice, expressed in radians or in degrees, verify the three basic identities.

Various answers.

3. Use the appropriate basic identity to calculate

a) $\sin t$, knowing that $\cos t = \frac{3}{5}$ and $270° \leqslant t \leqslant 360°$.

$$\sin^2 t = 1 - \cos^2 t = \frac{16}{25} \Rightarrow \sin t = \frac{-4}{5}$$

b) $\cos t$, knowing that $\sin t = \frac{40}{41}$ and $90° \leqslant t \leqslant 180°$.

$$\cos^2 t = 1 - \sin^2 t = \frac{81}{1600} \Rightarrow \sec t = \frac{-9}{40}$$

c) $\tan t$, knowing that $\sec t = \frac{-5}{4}$ and $180° \leqslant t \leqslant 270°$.

$$\tan^2 t = \sec^2 t - 1 = \frac{9}{16} \Rightarrow \tan t = \frac{3}{4}$$

d) $\cot t$, knowing that $\csc t = -\frac{13}{12}$ and $270° \leqslant t \leqslant 360°$.

$$\cot^2 t = \csc^2 t - 1 = \frac{25}{144} \Rightarrow \cot t = \frac{-5}{12}$$

e) $\sec t$, knowing that $\tan t = \frac{7}{24}$ and $180° \leqslant t \leqslant 270°$.

$$\sec^2 t = 1 + \tan^2 t = \frac{625}{576} \Rightarrow \sec t = \frac{-25}{24}$$

f) $\csc t$, knowing that $\cot t = \frac{4}{3}$ and $0° \leqslant t \leqslant 90°$.

$$\csc^2 t = 1 + \cot^2 t = \frac{25}{9} \Rightarrow \cot t = \frac{5}{3}$$

4. Reduce the following expressions to a single term.

a) $1 - \sin^2 t$ ___ $\cos^2 t$

b) $\sec^2 t - \tan^2 t$ ___ 1

c) $\cot^2 t - \csc^2 t$ ___ -1

d) $\sin t \sec t$ ___ $\tan t$

e) $\tan x \cdot \csc x$ ___ $\sec x$

f) $(1 - \sin^2 x) \sec^2 x$ ___ 1

g) $(1 + \tan^2 x) \sin^2 x$ ___ $\tan^2 x$

h) $\csc^2 x(1 - \cos^2 x)$ ___ 1

i) $(\sec^2 x - 1) \cot^2 x$ ___ 1

j) $\csc^2 x - \cot^2 x - \sin^2 x$ ___ $\cos^2 x$

5. Express each of the following trigonometric ratios in terms of $\sin x$ knowing that $0 \leqslant x \leqslant \frac{\pi}{2}$.

a) $\cos x$ ___ $\sqrt{1 - \sin^2 x}$

b) $\tan x$ ___ $\frac{\sin x}{\sqrt{1 - \sin^2 x}}$

c) $\cot x$ ___ $\frac{\sqrt{1 - \sin^2 x}}{\sin x}$

d) $\sec x$ ___ $\frac{1}{\sqrt{1 - \sin^2 x}}$

6. If $\sin t = 0.6$ and $\frac{\pi}{2} \leqslant t \leqslant \pi$, deduce the other 5 trigonometric ratios.

$\cos t = -0.8$, $\tan t = -\frac{3}{4}$, $\cot t = -\frac{4}{3}$, $\sec t = -\frac{5}{4}$, $\csc t = \frac{5}{3}$.

7. If $\cos t = \frac{12}{13}$ and $\frac{3\pi}{2} \leqslant t \leqslant 2\pi$, deduce the other 5 trigonometric ratios.

$\sin t = \frac{-5}{13}$, $\tan t = \frac{-5}{12}$, $\cot t = \frac{-12}{5}$, $\sec t = \frac{13}{12}$, $\csc t = \frac{-13}{5}$.

8. If $\tan t = \frac{3}{4}$ and $0 \leqslant t \leqslant \frac{\pi}{2}$, deduce the other 5 trigonometric ratios.

$\sec t = \frac{5}{4}$, $\cos t = \frac{4}{5}$, $\sin t = \frac{3}{5}$, $\cot t = \frac{4}{3}$, $\csc t = \frac{5}{3}$.

9. If $\cot t = \frac{-5}{12}$ and $\frac{3\pi}{2} \leqslant t \leqslant 2\pi$, deduce the other 5 trigonometric ratios.

$\csc t = \frac{-13}{12}$, $\sin t = \frac{-12}{13}$, $\cos t = \frac{5}{13}$, $\tan t = \frac{-12}{5}$, $\sec t = \frac{13}{5}$.

10. Simplify the following expressions.

a) $\frac{\sin^2 x + \cos^2 x}{1 - \cos^2 x}$ ___ $\csc^2 x$

b) $\frac{1 + \tan^2 x}{1 + \cot^2 x}$ ___ $\tan^2 x$

c) $\frac{\sec^2 x - \tan^2 x}{1 - \sin^2 x}$ ___ $\sec^2 x$

d) $\frac{\sec^2 x - 1}{\csc^2 x} \cdot \frac{\cot^2 x}{\sin^2 x}$ ___ 1

11. Simplify the following expressions.

a) $\frac{1 - \cos^2 x}{1 - \sin^2 x}$ ___ $\tan^2 x$

b) $\frac{1 + \tan^2 x}{1 + \cot^2 x}$ ___ $\tan^2 x$

c) $(1 + \sec x)(1 - \sec x)$ ___ $-\tan^2 x$

d) $\frac{\csc^2 x - \cot^2 x}{\cos^2 x}$ ___ $\sec^2 x$

12. Perform the following operations.

a) $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} =$ ___ $2 \sec^2 x$

b) $\frac{\cos x}{\sec x + 1} + \frac{\cos x}{\sec x - 1} =$ ___ $\frac{2}{\tan^2 x}$

ACTIVITY 2 Proving a trigonometric identity

Justify the steps which prove the identity: $\sin x + \cos x \cot x = \csc x$ according to the following procedures.

1st procedure: Transform the left side using algebraic manipulations to make it identical to the right side.

Steps	Justifications
$\sin x + \cos x \cot x = \csc x$	
1. $\sin x + \cos x \cdot \frac{\cos x}{\sin x} = \csc x$	*$\cot x = \frac{\cos x}{\sin x}$*
2. $\frac{\sin^2 x + \cos^2 x}{\sin x} = \csc x$	*Reducing to a common denominator.*
3. $\frac{1}{\sin x} = \csc x$	*$\sin^2 x + \cos^2 x = 1$*
4. $\csc x = \csc x$	*$\frac{1}{\sin x} = \csc x$*

2nd procedure: Transform the right side using algebraic manipulations to make it identical to the left side.

Steps	Justifications
$\sin x + \cos x \cot x = \csc x$	
1. $\sin x + \cos x \cot x = \frac{1}{\sin x}$	*$\csc x = \frac{1}{\sin x}$*
2. $\sin x + \cos x \cot x = \frac{\sin^2 x + \cos^2 x}{\sin x}$	*$1 = \sin^2 x + \cos^2 x$*
3. $\sin x + \cos x \cot x = \sin x + \frac{\cos^2 x}{\sin x}$	*Divide each term by sin x.*
4. $\sin x + \cos x \cot x = \sin x + \cos x \cot x$	*$\frac{\cos^2 x}{\sin x} = \cos x \cdot \frac{\cos x}{\sin x}$ and $\frac{\cos x}{\sin x} = \cot x$*

PROVING A TRIGONOMETRIC IDENTITY

To prove an identity, we simplify one side (usually the more complex side) to make it identical to the other side. The algebraic manipulations used consist of

- substituting expressions by other known identities.
- using the definitions of the trigonometric ratios.
- multiplying or dividing by the same trigonometric expression.
- factoring.
- reducing to a common denominator.

...

Ex.: See activity 2.

13. Prove the following trigonometric identities.

a) $\cot^2 x \sin^2 x + \sin^2 x = 1$

$\cot^2 x \sin^2 x + \sin^2 x = \frac{\cos^2 x}{\sin^2 x} \cdot \sin^2 x + \sin^2 x = \cos^2 x + \sin^2 x = 1$

b) $\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = 1$

$\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = \sin^2 x + \cos^2 x = 1$

c) $\frac{1 + \tan^2 x}{\cot^2 x + 1} = \tan^2 x$

$\frac{1 + \tan^2 x}{\cot^2 x + 1} = \frac{\sec^2 x}{\csc^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$

d) $\frac{\cos^2 x \cdot \tan x}{\cot x} - 1 = -\cos^2 x$

$\frac{\cos^2 x \tan x}{\cot x} - 1 = \frac{\cos^2 x \cdot \sin x \cdot \sin x}{\cos x \cdot \cos x} - 1 = \sin^2 x - 1 = \sin^2 x - (\sin^2 x + \cos^2 x) = -\cos^2 x$

e) $\frac{\sec x}{\cos x} - 1 = \tan^2 x$

$\frac{\sec x}{\cos x} - 1 = \frac{1}{\cos^2 x} - 1 = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$

f) $(1 + \cot^2 x)(1 - \cos^2 x) = 1$

$(1 + \cot^2 x)(1 - \cos^2 x) = \csc^2 x \cdot \sin^2 x = \frac{\sin^2 x}{\sin^2 x} = 1$

g) $2 \cos^2 a - 1 = 1 - 2 \sin^2 a$

$2 \cos^2 a - 1 = 2(1 - \sin^2 a) - 1 = 2 - 2 \sin^2 a - 1 = 1 - 2 \sin^2 a$

h) $\tan x + \cot x = \sec x \csc x$

$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x} = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \csc x$

14. Prove the following trigonometric identities.

a) $\sec x - \cos x = \sin x \tan x$

$\sin x \tan x = \sin x \cdot \frac{\sin x}{\cos x} = \frac{\sin^2 x}{\cos x} = \frac{1 - \cos^2 x}{\cos x} = \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} = \sec x - \cos x$

b) $\frac{1 + \sin x}{\cos x} = \frac{\cos x}{1 - \sin x}$

$\frac{1 + \sin x}{\cos x} = \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} = \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} = \frac{\cos^2 x}{\cos x(1 - \sin x)} = \frac{\cos x}{1 - \sin x}$

c) $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

$\frac{\sin^2 x}{1 - \cos x} = \frac{1 - \cos^2 x}{1 - \cos x} = \frac{(1 + \cos x)(1 - \cos x)}{1 - \cos x} = 1 + \cos x$

d) $(1 + \tan x)^2 + (1 - \tan x)^2 = 2 \sec^2 x$

$(1 + \tan x)^2 + (1 - \tan x)^2 = 1 + 2 \tan x + \tan^2 x + 1 - 2 \tan x + \tan^2 x = 2(1 + \tan^2 x) = 2 \sec^2 x$

e) $(1 + \tan x)(1 - \tan x) - (1 + \cot x)(1 - \cot x) = \frac{1 - \tan^4 x}{\tan^2 x}$

$(1 + \tan x)(1 - \tan x) - (1 + \cot x)(1 - \cot x) = 1 - \tan^2 x - (1 - \cot^2 x)$

$= \cot^2 x - \tan^2 x = \frac{1}{\tan^2 x} - \tan^2 x = \frac{1 - \tan^4 x}{\tan^2 x}$

5.9 Trigonometric equations

ACTIVITY 1 Solving a trigonometric equation

A marble is hanging at the end of a spring that oscillates over a table. The height h (in cm) of the marble, relative to the table, as a function of elapsed time t (in sec) since the start of the movement is given by $h = 10 \sin \pi t + 15$.

a) What is the period of the function that describes the movement of the spring? ***$p = 2$ s***

b) Justify the steps in solving the equation that enables you to calculate at what moments, during the first period of the movement, the marble is located 20 cm above the table.

1. $10 \sin \pi t + 15 = 20$ — ***Set up the equation.***
2. $10 \sin \pi t = 5$ — ***Subtract 15 from each side.***
3. $\sin \pi t = 0.5$ — ***Divide each side by 10.***
4. $\pi t = \frac{\pi}{6}$ or $\pi t = \frac{5\pi}{6}$ — ***Deduce the 2 possible values of πt.***
5. $t = \frac{1}{6}$ or $t = \frac{5}{6}$ — ***Deduce the 2 possible values of t.***

c) Keeping the periodicity of the movement in mind, determine the moments during the first 5 seconds when the marble is located 20 cm above the table.

$\frac{1}{6}$ s, $\frac{5}{6}$ s, $\frac{13}{6}$ s, $\frac{17}{6}$ s, $\frac{25}{6}$ s, $\frac{29}{6}$ s.

ACTIVITY 2 Solving more complex trigonometric equations

Consider the equation $3 \sin \theta - 2 \cos^2 \theta = 0$ where $\theta \in [0, 2\pi[$.

a) Express the non-zero side in terms of only $\sin \theta$.

$3 \sin \theta - 2(1 - \sin^2 \theta) = 0$

$2 \sin^2 \theta + 3 \sin \theta - 2 = 0$

b) Factor the non-zero side and then apply the zero product principle to determine the possible values of $\sin \theta$.

$(2 \sin \theta - 1)(\sin \theta + 2) = 0$

$\Leftrightarrow 2 \sin \theta - 1 = 0$ or $\sin \theta + 2 = 0$

$\Leftrightarrow \sin \theta = \frac{1}{2}$ or $\sin \theta = -2$

c) Of the two values found for $\sin \theta$, indicate which one must be rejected and explain why.

$\sin \theta = -2$ is rejected since $\forall \theta, -1 \leq \sin \theta \leq 1$

d) What are the solutions to the equation?

$\sin \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$.

TRIGONOMETRIC EQUATIONS

Let us illustrate the procedure for solving: $\tan^2\theta + 3\sec\theta\tan\theta - \sec^2\theta = 1$.

	$\tan^2\theta + 3\sec\theta\tan\theta - \sec^2\theta = 1$		
1	$\frac{\sin^2\theta}{\cos^2\theta} + \frac{3\sin\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta} = 1$	1	Use the definitions of the trigonometric ratios.
2	$\cos^2\theta \neq 0 \Leftrightarrow \cos\theta \neq 0$ $\theta \neq \frac{\pi}{2} + 2\pi n$ and $\theta \neq \frac{3\pi}{2} + 2\pi n,\ n \in \mathbb{Z}$	2	Set the restrictions on θ.
3	$\sin^2\theta + 3\sin\theta - 1 = \cos^2\theta$	3	Multiply each side by $\cos^2\theta$.
4	$\sin^2\theta + 3\sin\theta - 1 = 1 - \sin^2\theta$ $2\sin^2\theta + 3\sin\theta - 2 = 0$	4	Use identities so that the equation uses the same trigonometric ratio ($\sin\theta$ here).
5	$(2\sin\theta - 1)(\sin\theta + 2) = 0$	5	Factor the non-zero side.
6	$2\sin\theta - 1 = 0$ or $\sin\theta + 2 = 0$	6	Apply the zero product principle.
7	$\sin\theta = \frac{1}{2}$ or $\sin\theta = -2$ (reject)	7	Deduce the values for $\sin\theta$ and reject some, if necessary.
8	$\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$	8	Deduce the value of θ over $[0, p[$ where $p = 2\pi$ is the period of $\sin\theta$.
9	$S = \left\{\frac{\pi}{6} + 2\pi n\right\} \cup \left\{\frac{5\pi}{6} + 2\pi n\right\},\ n \in \mathbb{Z}$	9	Deduce the solution set S over $\mathbb{R}$, from the resulting solutions and period.

1. Solve the following trigonometric equations over $[0, 2\pi[$.

a) $\cos x \cdot (2\sin x + 1) = 0$ **$S = \left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$**

b) $(\sin x - 2)(2\cos x - 1) = 0$ **$S = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$**

2. Solve the following trigonometric equations over $\mathbb{R}$.

a) $(2\sin x + 1)(\sin x - 1) = 0$ **$S = \left\{\frac{5\pi}{6} + 2\pi n\right\} \cup \left\{\frac{7\pi}{6} + 2\pi n\right\} \cup \left\{\frac{\pi}{2} + 2\pi n\right\}$**

b) $(3\sin x - 1) \cdot \cos x = 0$ **$S = \{0{,}34 + 2\pi n\} \cup \{2{,}8 + 2\pi n\} \cup \left\{\frac{\pi}{2} + 2\pi n\right\}$**

3. Solve the following trigonometric equations over $[0, 2\pi[$.

a) $4\cos^2\theta - 1 = 0$

$S = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

b) $2\sin^2\theta - \sin\theta - 1 = 0$

$S = \left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

c) $\tan^2\theta - 3 = 0$

$S = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

d) $2\sin^2 2x - \sin 2x = 0$

$S = \left\{0, \frac{\pi}{12}, \frac{5\pi}{12}, \pi, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$

e) $2\sin^2\theta - \cos\theta - 1 = 0$

$S = \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

f) $\sec^2\theta + \sec\theta - 2 = 0$

$S = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

g) $\sec^2\theta - \tan\theta\sec\theta - 2 = 0$

$S = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

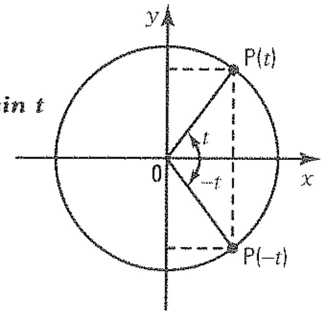
h) $4\tan^2 2\theta = \sec^2 2\theta$

$S = \left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

5.10 Trigonometric formulas

ACTIVITY 1 Opposite angle formulas

The trigonometric points P(t) and P($-t$) are symmetrical about the x-axis.



a) Compare

1. cos ($-t$) and cos t ***cos (−t) = cos t***
2. sin ($-t$) and sin t ***sin (−t) = −sin t***

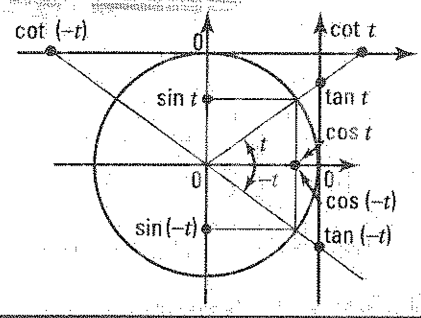
b) Prove that, $\forall\, t \in \mathbb{R}$,

1. tan ($-t$) = −tan t $\quad \tan(-t) = \dfrac{\sin(-t)}{\cos(-t)} = \dfrac{-\sin t}{\cos t} = -\tan t$
2. cot ($-t$) = −cot t $\quad \cot(-t) = \dfrac{\cos(-t)}{\sin(-t)} = \dfrac{\cos t}{-\sin t} = -\dfrac{\cos t}{\sin t} = -\cot t$
3. sec ($-t$) = sec t $\quad \sec(-t) = \dfrac{1}{\cos(-t)} = \dfrac{1}{\cos t} = \sec t$
4. csc ($-t$) = −csc t $\quad \csc(-t) = \dfrac{1}{\sin(-t)} = \dfrac{-1}{\sin t} = -\csc t$

OPPOSITE ANGLE FORMULAS

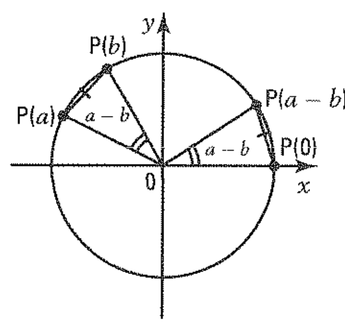
For any real t, we have:

$\sin(-t) = -\sin t$;	$\cos(-t) = \cos t$
$\tan(-t) = -\tan t$;	$\cot(-t) = -\cot t$
$\csc(-t) = -\csc t$;	$\sec(-t) = \sec t$



ACTIVITY 2 Addition formulas

On the right, we have represented the points P(a), P(b), P($a - b$) and P(0) on the trigonometric circle.



a) The following steps enable you to prove the formula:
$\cos(a - b) = \cos a \cos b + \sin a \sin b$.

1. Calculate $d(\text{P}(a), \text{P}(b))^2$.

We have: P(a) = (cos a, sin a), P(b) = (cos b, sin b) and

$d(P(a), P(b))^2 = (\cos b - \cos a)^2 + (\sin b - \sin a)^2$

$= \cos^2 b - 2 \cos a \cos b + \cos^2 a + \sin^2 b - 2 \sin a \sin b + \sin^2 b$

$= 2 - 2 \cos a \cos b - 2 \sin a \sin b$

2. Calculate $d(\text{P}(a - b), \text{P}(0))^2$.

We have: P(a − b) = (cos (a − b), sin (a − b)) and P(0) = (1, 0)

$d(P(a - b), P(0)) = [1 - \cos(a - b)]^2 + [\sin(a - b)]^2$

$= 1 - 2\cos(a - b) + \cos^2(a - b) + \sin^2(a - b)$

$= 2 - 2\cos(a - b)$

3. Knowing that $d(P(a), P(b)) = d(P(a-b), P(0))$, justify the formula.

We deduce that 2 – 2 cos a cos b – 2 sin a sin b = 2 – 2 cos (a – b),

that is cos (a – b) = cos a cos b + sin a sin b.

b) Use the formula: $\cos(a-b) = \cos a \cos b + \sin a \sin b$, and the opposite angle formulas (Activity 1) to prove the formula:

$\cos(a+b) = \cos a \cos b - \sin a \sin b$.

cos (a + b) = cos (a – (–b)) = cos a cos (–b) + sin a sin (–b) = cos a cos b – sin a sin b

c) Justify the steps in proving the formula: $\sin(a+b) = \sin a \cos b + \sin b \cos a$.

Steps	Justifications
1. $\sin(a+b) = \cos\left[\frac{\pi}{2} - (a+b)\right]$	***The sine of an angle is equal to the cosine of its complementary angle.***
2. $= \cos\left[\left(\frac{\pi}{2} - a\right) - b\right]$	***Distributive property***
3. $= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b$	***cos (x – y) = cos x cos y + sin x sin y.***
4. $= \sin a \cos b + \sin b \cos a$	***When 2 angles are complementary, the sine of one is equal to the cosine of the other.***

d) Use the formula: $\sin(a+b) = \sin a \cos b + \sin b \cos a$ and the opposite angle formulas to prove the formula: $\sin(a-b) = \sin a \cos b - \sin b \cos a$.

sin (a – b) = sin (a + (–b)) = sin a cos (–b) + sin (–b) cos a = sin a cos b – sin b cos a

e) Justify the steps in proving the formula: $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$.

Steps	Justifications
1. $\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)}$	***Definition of tangent.***
2. $= \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$	***Expansion of sin (a + b) see d).*** ***Expansion of cos (a + b) see b).***
3. $= \frac{(\sin a \cos b + \sin b \cos a) \div \cos a \cos b}{(\cos a \cos b - \sin a \sin b) \div \cos a \cos b}$	$\frac{x}{y} = \frac{x \div k}{y \div k}\ (\forall\, k \neq 0)$
4. $= \frac{\tan a + \tan b}{1 - \tan a \tan b}$	***Reduction and definition of tangent.***

f) Use the formula: $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$ and the opposite angle formulas to prove the formula:

$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$. ***tan (a – b) = tan (a + (–b))*** $= \frac{\tan a + \tan(-b)}{1 - \tan a \cdot \tan(-b)} = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

ADDITION FORMULAS

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

The opposite angle formulas enable you to deduce that

$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$a - b = a + (-b)$

ACTIVITY 3 Double-angle formulas

a) Knowing that $2a = a + a$, use the addition formulas to show that

1. $\sin 2a = 2 \sin a \cos a$.

   $\sin 2a = \sin(a + a) = \sin a \cos a + \sin a \cos a = 2 \sin a \cos a$

2. $\cos 2a = \cos^2 a - \sin^2 a$.

   $\cos 2a = \cos(a + a) = \cos a \cos a - \sin a \sin a = \cos^2 a - \sin^2 a$

3. $\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$ ***$\tan 2a = \tan(a + a) = \frac{\tan a + \tan a}{1 - \tan a \tan a} = \frac{2 \tan a}{1 - \tan^2 a}$***

b) Knowing that $\sin^2 a + \cos^2 a = 1$, prove that

1. $\cos 2a = 2 \cos^2 a - 1$.

   $\cos 2a = \cos^2 a - \sin^2 a = \cos^2 a - (1 - \cos^2 a) = 2 \cos^2 a - 1$

2. $\cos 2a = 1 - 2 \sin^2 a$.

   $\cos 2a = \cos^2 a - \sin^2 a = (1 - \sin^2 a) - \sin^2 a = 1 - 2 \sin^2 a$

DOUBLE-ANGLE FORMULAS

$$\sin 2a = 2 \sin a \cos a$$
$$\cos 2a = \cos^2 a - \sin^2 a$$
$$\ldots = 2 \cos^2 a - 1$$
$$\ldots = 1 - 2 \sin^2 a$$
$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

ACTIVITY 4 Supplementary angle formulas

a) Use the addition formulas to show that

1. $\sin(\pi - x) = \sin x$.

$$\sin(\pi - x) = \sin\pi\cos x - \sin x\cos\pi$$
$$= 0 \cdot \cos x - \sin x \cdot (-1)$$
$$= \sin x$$

2. $\cos(\pi - x) = -\cos x$.

$$\cos(\pi - x) = \cos\pi\cos x + \sin x\sin\pi$$
$$= (-1) \cdot \cos x + 0 \cdot \sin x$$
$$= -\cos x$$

3. $\tan(\pi - x) = -\tan x$.

$$\tan(\pi - x) = \frac{\tan\pi - \tan x}{1 + \tan\pi\tan x} = \frac{0 - \tan x}{1 + 0 \cdot \tan x} = -\tan x$$

b) Deduce that

1. $\sec(\pi - x) = -\sec x$.

$$\sec(\pi - x) = \frac{1}{\cos(\pi - x)} = \frac{1}{-\cos x} = -\frac{1}{\cos x} = -\sec x$$

2. $\csc(\pi - x) = \csc x$.

$$\csc(\pi - x) = \frac{1}{\sin(\pi - x)} = \frac{1}{\sin x} = \csc x$$

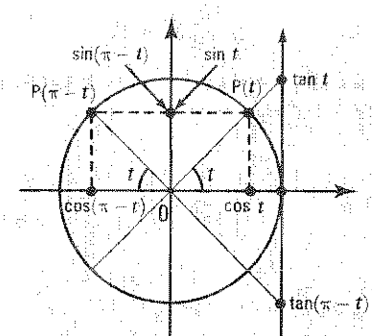
3. $\cot(\pi - x) = -\cot x$.

$$\cot(\pi - x) = \frac{1}{\tan(\pi - x)} = \frac{1}{-\tan x} = -\frac{1}{\tan x} = -\cot x$$

SUPPLEMENTARY ANGLE FORMULAS

$$\sin(\pi - t) = \sin t$$
$$\cos(\pi - t) = -\cos t$$
$$\tan(\pi - t) = -\tan t$$

$$\sec(\pi - t) = -\sec t$$
$$\csc(\pi - t) = \csc t$$
$$\cot(\pi - t) = -\cot t$$



ACTIVITY 5 Complementary angle formulas

a) Use the addition formulas to show that

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \sin x \cos\frac{\pi}{2}$

$= 1 \cdot \cos x - \sin x \cdot 0$

$= \cos x$

2. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 \cdot \cos x + 1 \sin x$

$= \sin x$

b) Deduce that

1. $\tan\left(\frac{\pi}{2} - x\right) = \cot x$.

$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x} = \cot x$

2. $\sec\left(\frac{\pi}{2} - x\right) = \csc x$.

$\sec\left(\frac{\pi}{2} - x\right) = \frac{1}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{1}{\sin x} = \csc x$

3. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$.

$\csc\left(\frac{\pi}{2} - x\right) = \frac{1}{\sin\left(\frac{\pi}{2} - x\right)} = \frac{1}{\cos x} = \sec x$

4. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$.

$\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)} = \frac{\sin x}{\cos x} = \tan x$

COMPLEMENTARY ANGLE FORMULAS

$\sin\left(\frac{\pi}{2} - t\right) = \cos t$

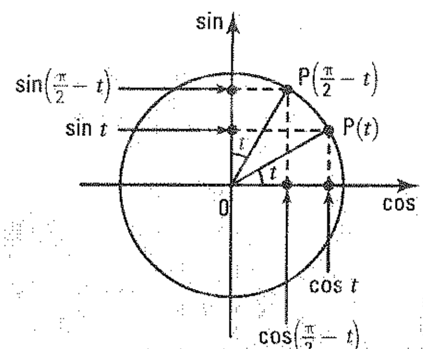
$\cos\left(\frac{\pi}{2} - t\right) = \sin t$

$\tan\left(\frac{\pi}{2} - t\right) = \cot t$

$\sec\left(\frac{\pi}{2} - t\right) = \csc t$

$\csc\left(\frac{\pi}{2} - t\right) = \sec t$

$\cot\left(\frac{\pi}{2} - t\right) = \tan t$



1. Knowing that $a = 60°$ and $b = 30°$, verify that

a) $\sin(a + b) = \sin a \cos b + \sin b \cos a$

$\sin 90° = \sin 60° \cos 30° + \sin 30° \cos 60°$

$1 = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$

b) $\sin(a - b) = \sin a \cos b - \sin b \cos a$

$\sin 30° = \sin 60° \cos 30° - \sin 30° \cos 60°$

$\frac{1}{2} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$

c) $\cos(a + b) = \cos a \cos b - \sin a \sin b$

$\cos 90° = \cos 60° \cos 30° - \sin 60° \sin 30°$

$0 = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

d) $\cos(a - b) = \cos a \cos b + \sin a \sin b$

$\cos 30° = \cos 60° \cos 30° + \sin 60° \sin 30°$

$\frac{\sqrt{3}}{2} = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

2. Knowing that $a = 30°$, verify that

a) $\sin 2a = 2 \sin a \cos a$

$\sin 60° = 2 \sin 30° \cos 30°$

$\frac{\sqrt{3}}{2} = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$

b) $\cos 2a = \cos^2 a - \sin^2 a$

$\cos 60° = \cos^2 30° - \sin^2 30°$

$\frac{1}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

c) $\cos 2a = 2 \cos^2 a - 1$

$\cos 60° = 2 \cos^2 30° - 1$

$\frac{1}{2} = 2 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - 1$

d) $\cos 2a = 1 - 2 \sin^2 a$

$\cos 60° = 1 - 2 \sin^2 30°$

$\frac{1}{2} = 1 - 2\left(\frac{1}{2}\right)^2$

3. Use the addition formulas to simplify

a) $\sin(\pi + x) = \sin \pi \cos x + \sin x \cos \pi = -\sin x$

b) $\cos(\pi + x) = \cos \pi \cos x - \sin \pi \sin x = -\cos x$

c) $\sin\left(\frac{\pi}{2} + x\right) = \sin \frac{\pi}{2} \cos x + \sin x \cos \frac{\pi}{2} = \cos x$

d) $\cos\left(\frac{\pi}{2} + x\right) = \cos \frac{\pi}{2} \cos x - \sin \frac{\pi}{2} \sin x = -\sin x$

4. Knowing that $\sin a = \frac{3}{5}$ and $\sin b = \frac{12}{13}$ and that $0 \leqslant a \leqslant \frac{\pi}{2}$ and $0 \leqslant b \leqslant 2$, calculate

a) $\sin(a + b)$

$\sin a = \frac{3}{5}$ and $0 \leqslant a \leqslant \frac{\pi}{2} \Rightarrow \cos a = \frac{4}{5}$; $\sin b = \frac{12}{13}$ and $0 \leqslant b \leqslant \frac{\pi}{2} \Rightarrow \cos b = \frac{5}{13}$

$\sin(a + b) = \sin a \cos b + \sin b \cos a = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$

b) $\cos(a + b)$

$\cos(a + b) = \cos a \cos b - \sin a \sin b = \frac{20}{65} - \frac{36}{65} = \frac{-16}{65}$

c) $\tan(a + b)$

$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \cdot \frac{12}{5}} = \frac{\frac{63}{20}}{1 - \frac{36}{20}} = \frac{-63}{16}$

d) $\sin 2a$

$\sin 2a = 2 \sin a \cos a = \frac{24}{25}$

e) $\cos 2a$

$\cos 2a = \cos^2 a - \sin^2 a = \frac{7}{25}$

5. Prove the following identities.

a) $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$ $\frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \tan x}{\sec^2 x} = 2 \tan x \cdot \cos^2 x = \frac{2 \sin x}{\cos x} \cdot \cos^2 x = 2 \sin x \cos x = \sin 2x$

b) $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$ $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1 - \tan^2 x}{\sec^2 x} = (1 - \tan^2 x) \cdot \cos^2 x = \cos^2 x - \sin^2 x = \cos 2x$

c) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ $\frac{2 \tan x}{1 - \tan^2 x} = \frac{\frac{2 \sin x}{\cos x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{2 \sin x}{\cos x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x} \cdot \frac{\sin 2x}{\cos 2x} = \tan 2x$

6. Show that $\tan 75°$ has the exact value: $2 + \sqrt{3}$.

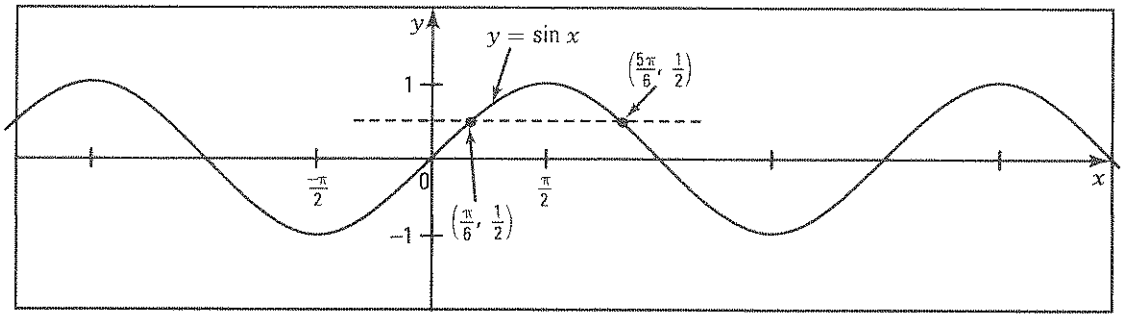
$\frac{\tan 45° + \tan 30°}{1 - \tan 45° \cdot \tan 30°} = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})^2}{(3 - \sqrt{3})(3 + \sqrt{3})} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$

5.11 Inverse trigonometric functions

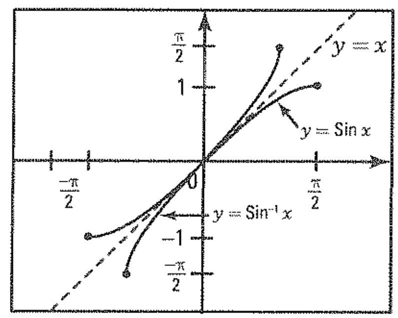
ACTIVITY 1 Arcsine function

a) The sine function is represented below. Explain why the inverse of the sine function is not a function.

There exists a horizontal line that passes through the curve at more than one point.



b) A portion of the sine function, when the variable x varies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is represented on the right.



This new graph represents the function called **principal sine** denoted Sin.

1. Determine
   1) dom Sin $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 2) ran Sin ***[−1, 1]***
2. Is the inverse of the Sin function a function? ***Yes***
3. Deduce the graph of the inverse Sin^{-1} by a reflection about the line $y = x$.
4. Complete
   $\text{Sin}\,\frac{\pi}{6} = \frac{1}{2} \Leftrightarrow \text{Sin}^{-1}\,\frac{1}{2} =$ $\frac{\pi}{6}$
5. Determine
   1) dom Sin^{-1} ***[−1, 1]*** 2) ran Sin^{-1} $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

ARCSINE FUNCTION

- The **principal sine** function, denoted Sin, is defined by

  $\text{Sin}: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

  $x \mapsto y = \text{Sin}\,x$

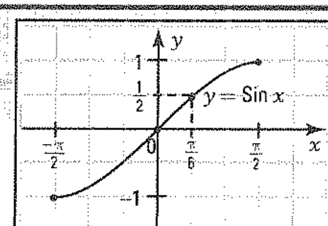
- The inverse Sin^{-1} of the principal sine function is a function called **arcsine function**, denoted arcsin.

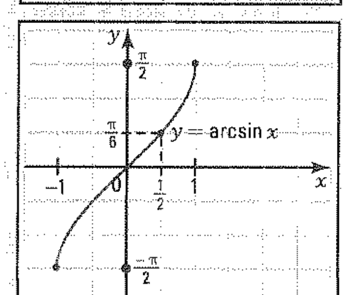
  $\text{arcsin}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  $x \mapsto y = \text{arcsin}\,x$

  Ex.: $\text{arcsin}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ since $\text{Sin}\,\frac{\pi}{6} = \frac{1}{2}$

- We have: dom arcsin $= [-1, 1]$ and ran arcsin $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

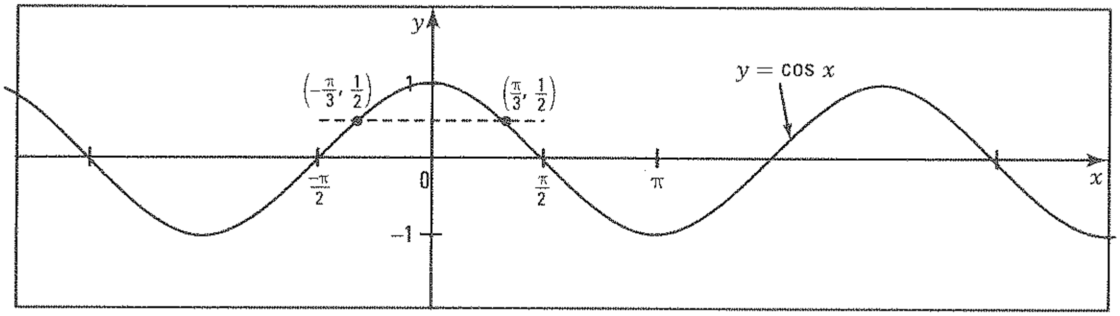




ACTIVITY 2 Arccosine function

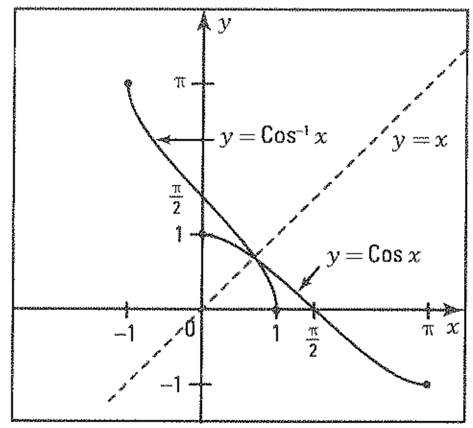
a) The cosine function is represented below. Explain why the inverse of the cosine function is not a function.

There exists a horizontal line that passes through the curve at more than one point.

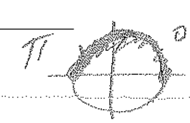


b) A portion of the cosine function, when the variable x varies between 0 and π is represented on the right.

This new graph represents the function called **principal cosine** denoted Cos.



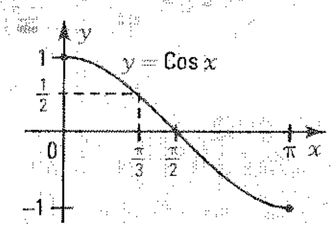
1. Determine
   1) dom Cos ***[0, π]*** 2) ran Cos ***[−1, 1]***
2. Is the inverse of the Cos function a function?
   Yes
3. Deduce the graph of the inverse Cos^{-1} by a reflection about the line $y = x$.
4. Complete
   $\text{Cos}\,\frac{\pi}{3} = \frac{1}{2} \Leftrightarrow \text{Cos}^{-1}\,\frac{1}{2} =$ ***$\frac{\pi}{3}$***



5. Determine
   1) dom Cos^{-1} ***[−1, 1]*** 2) ran Cos^{-1} ***[0, π]***

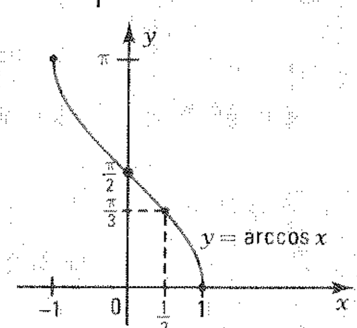
ARCCOSINE FUNCTION

- The principal cosine function, denoted Cos, is defined by
  Cos: $[0, \pi] \to [-1, 1]$
  $x \mapsto y = \text{Cos}\,x$

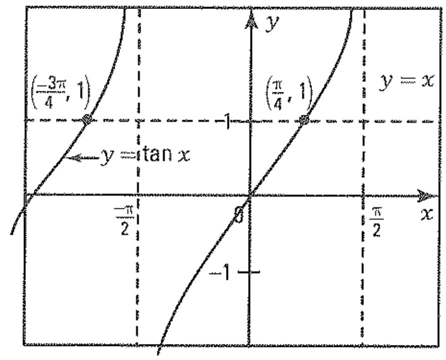


- The inverse Cos^{-1} of the principal cosine function is a function called **arccosine** function, denoted arccos.
  arccos: $[-1, 1] \to [0, \pi]$
  $x \mapsto y = \arccos x$

  Ex: $\arccos\frac{1}{2} = \frac{\pi}{3}$ since $\text{Cos}\,\frac{\pi}{3} = \frac{1}{2}$
- We have: dom arccos $= [-1, 1]$ and ran arccos $= [0, \pi]$.



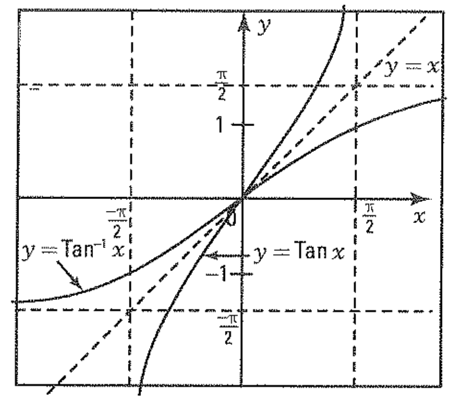
ACTIVITY 3 Arctangent function



a) The tangent function is represented below. Explain why the inverse of the tangent function is not a function.

There exists a horizontal line that passes through the curve at more than one point.

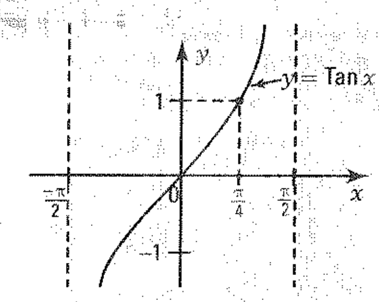
b) A portion of the tangent function, when the variable x varies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is represented on the right. This new graph represents the function called **principal tangent** denoted Tan.



1. Determine
   1) dom Tan $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ 2) ran Tan $\mathbb{R}$
2. Is the inverse of the Tan function a function?
   Yes
3. Deduce the graph of the inverse Tan^{-1} by a reflection about the line $y = x$.
4. Complete
   $\text{Tan}\, \frac{\pi}{4} = 1 \Leftrightarrow \text{Tan}^{-1}\, 1 =$ $\frac{\pi}{4}$
5. Determine
   1) dom Tan^{-1} $\mathbb{R}$ 2) ran Tan^{-1} $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$
6. Does the inverse Tan^{-1} have any asymptotes? If yes, give their equations.
   Yes, 2 horizontal asymptotes with equations $y = \frac{-\pi}{2}$ and $y = \frac{\pi}{2}$.

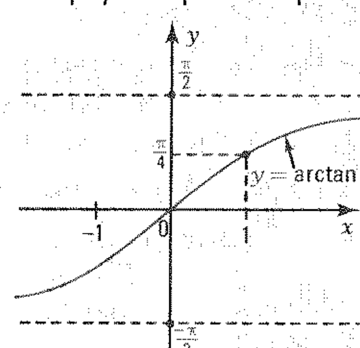
ARCTANGENT FUNCTION

- The **principal tangent** function, denoted Tan, is defined by
  $\text{Tan}: \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$
  $x \mapsto y = \text{Tan}\, x$

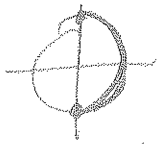
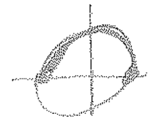


- The inverse Tan^{-1} of the principal tangent function is a function called **arctangent** function, denoted arctan.
  $\arctan: \mathbb{R} \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$
  $x \mapsto y = \arctan x$

  Ex.: $\arctan 1 = \frac{\pi}{4}$ since $\text{Tan}\, \frac{\pi}{4} = 1$



- We have: dom arctan $= \mathbb{R}$ and ran arctan $= \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
- The arctan function has 2 horizontal asymptotes:
  $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

sin⁻¹  cos⁻¹ 

1. Without using a calculator, determine the exact value of

a) $\sin^{-1}\frac{\sqrt{2}}{2}$ ___ $\frac{\pi}{4}$ b) $\cos^{-1}\frac{\sqrt{3}}{2}$ ___ $\frac{\pi}{6}$ c) $\tan^{-1}\sqrt{3}$ ___ $\frac{\pi}{3}$ d) $\arcsin\left(-\frac{1}{2}\right)$ ___ $-\frac{\pi}{6}$

e) $\arccos\left(-\frac{1}{2}\right)$ ___ $\frac{2\pi}{3}$ f) $\arctan\frac{\sqrt{3}}{3}$ ___ $\frac{\pi}{6}$ g) $\arcsin 1$ ___ $\frac{\pi}{2}$ h) $\arccos(-1)$ ___ π

2. Find the zeros of the following functions.

a) arcsine ___ *0* b) arccosine ___ *1* c) arctangent ___ *0*

3. Study the sign of the following functions.

a) arcsine ___ $\arcsin x \leqslant 0$ *if* $-1 \leqslant x \leqslant 0$ *and* $\arcsin x \geqslant 0$ *if* $0 \leqslant x \leqslant 1$

b) arccosine ___ $\arccos x \geqslant 0, \forall x \in [-1, 1]$

c) arctangent ___ $\arctan x \leqslant 0$ *if* $x \leqslant 0$ *and* $\arctan x \geqslant 0$ *if* $x \geqslant 0$

4. Calculate

a) $\sin\left(\cos^{-1}\frac{1}{2}\right)$ ___ $\frac{\sqrt{3}}{2}$ b) $\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$ ___ $\frac{1}{2}$ c) $\tan\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$ ___ 1

d) $\sin(\tan^{-1} 1)$ ___ $\frac{\sqrt{2}}{2}$ e) $\cos(\tan^{-1}\sqrt{3})$ ___ $\frac{1}{2}$ f) $\tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ ___ $-\sqrt{3}$

g) $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ ___ $\frac{\sqrt{3}}{2}$ h) $\sin^{-1}\left(\tan\frac{\pi}{4}\right)$ ___ $\frac{\pi}{2}$ i) $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$ ___ -1

5. True or false?

a) The arcsine function is always increasing. ___ ***True***

b) The arccosine function is always decreasing. ___ ***True***

c) The arctangent function is always increasing. ___ ***True***

d) The arcsine and arccosine functions have the same domain. ***True***

e) The arcsine and arccosine functions have the same range. ***False***

6. Solve the following equations.

a) $\arcsin x = \frac{\pi}{4}$ ___ $x = \frac{\sqrt{2}}{2}$ b) $\arccos x = \frac{5\pi}{6}$ ___ $x = \frac{-\sqrt{3}}{2}$ c) $\arctan x = \frac{-\pi}{4}$ ___ $x = -1$

d) $\arctan x = -\frac{\pi}{3}$ ___ $x = -\sqrt{3}$ e) $\arcsin x = \frac{-\pi}{4}$ ___ $x = -\frac{\sqrt{2}}{2}$ f) $\arccos x = \frac{\pi}{2}$ ___ $x = 0$

7. Solve the following equations.

a) $\sin(\arccos x) = \frac{1}{2}$ ___ $x = \frac{\sqrt{3}}{2}$ b) $\tan(\arcsin x) = 1$ ___ $x = \frac{\sqrt{2}}{2}$

c) $\cos(\arctan x) = 1$ ___ $x = 0$ d) $\sin(\arctan x) = \frac{-\sqrt{2}}{2}$ ___ $x = -1$

8. True or false?

a) $\arcsin x + \arccos x = \frac{\pi}{2}$ ___ ***True***

b) $\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$ ___ ***True***

Evaluation 5

1. a) Convert $\frac{5\pi}{36}$ into degrees. *25°*

b) Convert 75° into radians. $\frac{5\pi}{12}$ *rad*

2. A central angle measuring 54° sub-tends an arc on the circle.
If the circle has a radius of 12 cm, calculate the length of the sub-tended arc (round to the nearest tenth). *11.3 cm*

3. What are the coordinates of the trigonometric point P(t) located in the 3rd quadrant if $\cos t = -\frac{15}{17}$? $P(t) = \left(\frac{-15}{17}, \frac{-8}{17}\right)$

4. Evaluate $\sin\left(t - \frac{\pi}{4}\right)$ if $\frac{\pi}{2} < t < \pi$ and $\sin t = \frac{\sqrt{3}}{2}$.

$\cos t = -\frac{1}{2}$

$\sin\left(t - \frac{\pi}{4}\right) = \sin t \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos t = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$

5. Evaluate $\cos 2t$ if $\cos t = \frac{4}{5}$.

$\cos 2t = 2\cos^2 t - 1 = \frac{7}{25}$

6. Find the coordinates of the point P(t), located in the 3rd quadrant, if $\tan t = \frac{8}{15}$?

$\sec^2 t = 1 + \tan^2 t = \frac{289}{225}$; $\sec t = \frac{-17}{15}$; $\cos t = \frac{-15}{17}$; $\sin t = \frac{-8}{17} \Rightarrow P(t) = \left(\frac{-15}{17}, \frac{-8}{17}\right)$

7. If $P(t) = \left(-\frac{3}{5}, \frac{4}{5}\right)$, find the coordinates of the point P($2t$).

$P(t) = (\cos 2t, \sin 2t) = (\cos^2 t - \sin^2 t, 2 \sin t \cos t) = \left(\frac{-7}{25}, \frac{-24}{25}\right)$

8. Find the exact Cartesian coordinates of the trigonometric point $P\left(-\frac{19\pi}{6}\right)$.

$P\left(\frac{-19\pi}{6}\right) = P\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

9. Knowing that $\sin a = \frac{3}{5}$, $\cos a = \frac{4}{5}$, $\sin b = \frac{5}{13}$, $\cos b = \frac{12}{13}$, find the value of

a) $\sin(a + b)$ $\quad \sin a \cos b + \sin b \cos a = \frac{56}{65}$

b) $\cos(a - b)$ $\quad \cos a \cos b + \sin a \sin b = \frac{63}{65}$

c) $\tan(a + b)$ $\quad \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$

10. Simplify $\frac{\sin 2\theta}{1 + \cos 2\theta}$. $\quad \frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin\theta \cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$

11. Simplify $\frac{\sin 4a \cos 2a - \sin 2a \cos 4a}{\cos 2a \cos a + \sin 2a \sin a}$. $\quad \frac{\sin(4a - 2a)}{\cos(2a - a)} = \frac{\sin 2a}{\cos a} = \frac{2 \sin a \cos a}{\cos a} = 2 \sin a.$

12. Given the trigonometric point $P(t) = \left(\frac{3}{5}, \frac{4}{5}\right)$, find the coordinates of the following trigonometric points.

a) $P\left(t + \frac{\pi}{2}\right)$ $\left(-\frac{4}{5}, \frac{3}{5}\right)$ b) $P\left(t - \frac{\pi}{2}\right)$ $\left(\frac{4}{5}, -\frac{3}{5}\right)$

c) $P(t + \pi)$ $\left(-\frac{3}{5}, \frac{-4}{5}\right)$ d) $P(\pi - t)$ $\left(-\frac{3}{5}, \frac{4}{5}\right)$

13. If $P\left(\frac{a+2}{5}, \frac{a+1}{5}\right)$ is a trigonometric point located in the 3rd quadrant, determine a.

$\left(\frac{a+2}{5}\right)^2 + \left(\frac{a+1}{5}\right)^2 = 1$; $2a^2 + 6a - 20 = 0$; $a = -5$ or $a = 2$.

Reject the solution $a = 2$ since the trigonometric point is located in the 3rd quadrant.

14. If $\sin t = \frac{15}{17}$ and $\frac{\pi}{2} \leq t \leq \pi$, find

a) $\cos t$ $\frac{-8}{17}$ b) $\tan t$ $\frac{-15}{8}$ c) $\cot t$ $\frac{-8}{15}$ d) $\sec t$ $\frac{-17}{8}$ e) $\csc t$ $\frac{17}{15}$

15. The function f is given by the rule $f(x) = -4 \cos \frac{\pi}{6}(x - 2) + 2$. Determine

a) the amplitude: 4 b) the period of f. 12 c) ran f. $[-2, 6]$

d) the zeros of f. $\{4 + 12n\} \cup \{12 + 12n\}$

e) the sign of f over $[2, 14]$. $f(x) < 0 \Leftrightarrow x \in [2, 4[\cup]12, 14]$, $f(x) > 0 \Leftrightarrow x \in]4, 12[$

16. Determine the zeros of the following trigonometric functions over $\mathbb{R}$.

a) $f(x) = 2 \sin \frac{\pi}{2}(x - 3) - 1$ $\left\{\frac{10}{3} + 4n\right\} \cup \left\{\frac{14}{3} + 4n\right\}$

b) $f(x) = -6 \cos \frac{\pi}{3}(x + 1) + 3$ $\{0 + 6n\} \cup \{4 + 6n\}$

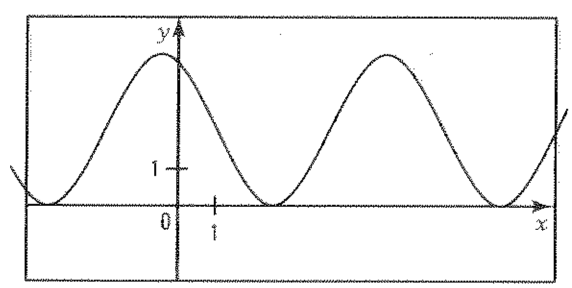
c) $f(x) = 2 \tan \frac{\pi}{4}(x - 1) + 2$ $\{4 + 8n\} \cup \{8 + 8n\}$

d) $f(x) = 5 \sin \frac{2\pi}{3}(x + 1) + 2$ $\{-1.2 + 3n\} \cup \{0.70 + 3n\}$

e) $f(x) = -2 \cos 2(x + 1) + 1$ $\{-0.48 + \pi n\} \cup \{1.62 + \pi n\}$

17. Find the rule of the sinusoidal function represented on the right.

$y = -2 \sin \frac{\pi}{3}(x - 1) + 2$



18. Solve the following trigonometric equations over $\mathbb{R}$.

a) $2 \sin x(\cos x - 1) = 0$ $S = \{0 + 2\pi n\} \cup \{\pi + 2\pi n\}$

b) $(2 \sin x + 1)(2 \cos x - 1) = 0$ $S = \left\{\frac{7\pi}{6} + 2\pi n\right\} \cup \left\{\frac{11\pi}{6} + 2\pi n\right\} \cup \left\{\frac{\pi}{3} + 2\pi n\right\} \cup \left\{\frac{5\pi}{3} + 2\pi n\right\}$

c) $2 \sin^2 x - 5 \sin x + 2 = 0$ $S = \left\{\frac{\pi}{6} + 2\pi n\right\} \cup \left\{\frac{5\pi}{6} + 2\pi n\right\}$

d) $\tan^2 x - 1 = 0$ $S = \left\{\frac{\pi}{4} + \pi n\right\} \cup \left\{\frac{3\pi}{4} + \pi n\right\}$

19. Solve the following equations over $[0, 2\pi[$.

a) $2 \sin^2 x - 1 = 0$ $S = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

b) $(2 \cos x + 1)(\sin x - 3) = 0$ $S = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

c) $25 \sin^2 x - 9 = 0$ $S = \{0.64; 2.50; 3.78; 5.64\}$

d) $(5 \cos x + 2)(2 \sin x - 1) = 0$ $S = \left\{1.98 ; 4.30 ; \frac{\pi}{6} ; \frac{5\pi}{6}\right\}$

20. True of false?

a) $\sin(a + b) = \sin a + \sin b$ **False**

b) $\sin 2a = 2 \sin a$ **False**

c) $\sin x^2 = \sin^2 x$ **False**

d) $\cos 2a = \cos^2 a - \sin^2 a$ **True**

21. A metal ball is attached to the end of a vertical spring. The spring is stretched to a length of 5 cm in relation to its equilibrium point and then released.

The function which gives the ball's position (in sec) relative to its equilibrium point, as a function of time t (in cm) since the release of the spring is given by the rule: $p(t) = 5 \cos \pi t$, $0 \leq t \leq 10$.

At what moments over the first 5 seconds is the spring's position even with its equilibrium point?

$p(t) = 0 \Leftrightarrow 5 \cos \pi t = 0 \Leftrightarrow \pi t = \frac{\pi}{2}$ or $\pi t = \frac{3\pi}{2} \Leftrightarrow t = \frac{1}{2}$ or $t = \frac{3}{2}$; period = 2 s.

The points are: 0.5 s; 1.5 s; 2.5 s; 3.5 s and 4.5 s.

22. The bird population of a certain region varies throughout the year. The population $P(t)$, in thousands, as a function of time t since January 1st is described by the rule: $P(t) = -6.4 \cos\left(\frac{\pi}{6} t\right) + 10$.

At what moments in the year are there 6800 birds in this region?

$-6.4 \cos\left(\frac{\pi}{6} t\right) + 10 = 6.8 \Leftrightarrow \cos \frac{\pi}{6} t = 0.5 \Leftrightarrow \frac{\pi t}{6} = \frac{\pi}{3}$ or $\frac{\pi t}{6} = \frac{5\pi}{3} \Leftrightarrow t = 2$ or $t = 10$.

March 1st or October 1st.

23. The current I (in amperes) produced by a generator is given by the rule $I = 20 \sin 30\pi t$ where t represents the time (in seconds) since the moment the generator was turned on. How long, after turning the generator on, do we observe an intensity of 10 amperes

a) the first time? *After $\frac{1}{180}$ seconds.*

b) the second time? *After $\frac{5}{180}$ seconds.*

24. Prove the following identities.

a) $\frac{1-\sin t}{\cos t} = \frac{\cos t}{1+\sin t}$

$\frac{1-\sin t}{\cos t} = \frac{(1-\sin t)(1+\sin t)}{\cos t\,(1+\sin t)} = \frac{1-\sin^2 t}{\cos t\,(1+\sin t)} = \frac{\cos^2 t}{\cos t\,(1+\sin t)} = \frac{\cos t}{1+\sin t}$

b) $\frac{\sin 2t}{1+\cos 2t} = \tan t$

$\frac{\sin 2t}{1+\cos 2t} = \frac{2\sin t\cos t}{1+(2\cos^2 t-1)} = \frac{2\sin t\cos t}{2\cos^2 t} = \frac{\sin t}{\cos t} = \tan t$

c) $\cos^4 t - \sin^4 t = \cos 2t$

$\cos^4 t - \sin^4 t = (\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t) = 1(\cos 2t) = \cos 2t$

d) $\frac{\sec t - 1}{\tan t} = \csc t - \cot t$

$\frac{\sec t - 1}{\tan t} = (\sec t - 1)\frac{\cos t}{\sin t} = \frac{1}{\sin t} - \frac{\cos t}{\sin t} = \csc t - \cot t$

25. Calculate.

a) $\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ $\frac{1}{2}$

b) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ $\frac{\sqrt{3}}{2}$

c) $\tan(\sin^{-1} 0)$ 0

d) $\cos\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$ $\frac{\sqrt{3}}{2}$

e) $\tan\left(\sin^{-1}\left(\frac{-1}{2}\right)\right)$ $-\frac{\sqrt{3}}{3}$

f) $\sec\left(\tan^{-1}\sqrt{3}\right)$ 2

26. Solve the following equations.

a) $\sin(\arccos x) = \frac{\sqrt{3}}{2}$ $x = \frac{1}{2}$

b) $\tan(\arcsin x) = -1$ $x = \frac{-\sqrt{2}}{2}$

c) $\sec(\arcsin x) = \frac{2}{\sqrt{3}}$ $x = \frac{1}{2}$

d) $\csc(\arctan x) = 2$ $x = \frac{\sqrt{3}}{3}$