Chapter 7

Conics

CHALLENGE 7

1. Name the set of points M in the plane

a) located at the same distance from a fixed point. ***Circle***

b) such that the sum of the distances to two fixed points is constant. ***Ellipse***

c) such that the absolute value of the difference of the distances to two fixed points is constant. ***Hyperbola***

d) located at the same distance to a fixed point and a fixed line. ***Parabola***

2. Determine the equation of the geometric locus defined by the set of points $M(x, y)$

a) located 5 units from the origin $0(0, 0)$. $x^2 + y^2 = 25$

b) such that the sum of the distances from point M to points $(-4, 0)$ and $(4, 0)$ is equal to 10.

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

c) such that the absolute value of the difference between the distances from point M to points $(-10, 0)$ and $(10, 0)$ is equal to 12.

$\frac{x^2}{36} - \frac{y^2}{64} = 1$

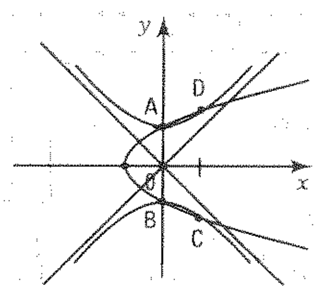
d) located at the same distance from the point $(-2, 3)$ and the line with equation $x = 4$.

$(x - 1) = -12(y - 3)^2$

3. a) Draw, in the Cartesian plane, the curve whose equation is $x^2 - y^2 = -1$ and the curve whose equation is $y^2 = x + 1$.

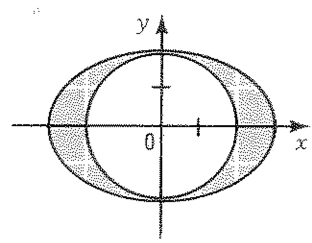
b) Find algebraically the intersection points of these two curves.

A(0, 1); B(0, −1); C(1, $-\sqrt{2}$); D(1, $+\sqrt{2}$)



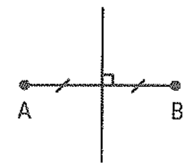
4. Represent, in the Cartesian plane, the set of points $M(x, y)$ such that

$$\begin{cases} \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \\ x^2 + y^2 \geq 4 \end{cases}$$



7.1 Geometric locus

ACTIVITY 1 Geometric locus



a) Given two distinct points A and B in a plane, what do we call the set of points whose distances from points A and B are equal?

The perpendicular bisector of the line segment AB.

b) The set of points having a common metric property is called a **geometric locus**. Give examples of geometric loci.

The perpendicular bisector of a line segment, the line passing through two given points, the bisector of an angle, ...

c) In the Cartesian plane, a locus is defined by an equation called **locus equation**.

Given points A(−1, 2) and B(3, 4), find the locus equation of the points whose distances from points A and B are equal.

M(1, 3) midpoint of $\overline{AB}$; $m = \frac{1}{2}$ (slope of $\overline{AB}$).

Locus equation (perpendicular bisector of $\overline{AB}$): $y = -2x + 5$.

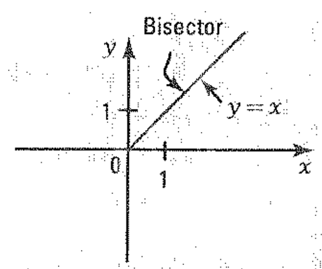
GEOMETRIC LOCUS

The set of points having a common characteristic is called a **geometric locus**.

In a Cartesian plane, the common property of the points (x, y) of a locus is translated by a two-variable equation called **locus equation**.

Ex.: The set of points in the 1st quadrant of the Cartesian plane whose distances from the x-axis and the y-axis are equal is a geometric locus corresponding to the bisector of the 1st quadrant.

The equation of the locus is: $y = x$ $(x \geqslant 0)$.



1. In each of the following cases, describe the geometric locus and give its equation if it is the set of points in the Cartesian plane

a) located 2 units from the x-axis and having a positive y-coordinate.

Horizontal line passing through (0, 2). $y = 2$

b) located 3 units from the y-axis and having a negative x-coordinate.

Vertical line passing through (−3, 0). $x = -3$

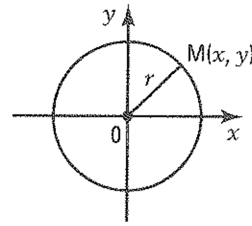
c) whose distances from the x-axis and the y-axis are equal.

The line with equation $y = x$ and the line with equation $y = -x$.

7.2 Circle

ACTIVITY 1 Circle centred at the origin

a) What is the definition of a circle of radius r centred at the origin?
The set of all points located at a distance r from 0.

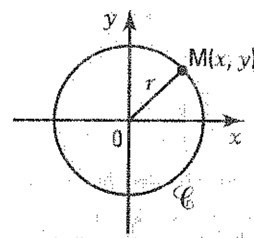


b) Justify the steps allowing us to find the equation of a circle $\mathscr{C}$ of radius r centred at $0(0, 0)$.

	Steps	Justifications
1.	$M(x, y) \in \mathscr{C} \Leftrightarrow d(0, M) = r$	*Definition of circle $\mathscr{C}$.*
2.	$\Leftrightarrow \sqrt{x^2 + y^2} = r$	*Formula for the distance between 0 and M.*
3.	$\Leftrightarrow x^2 + y^2 = r^2$	*When a and b are positive, we have the equivalence: $a = b \Leftrightarrow a^2 = b^2$.*

CIRCLE CENTRED AT THE ORIGIN

- A circle centred at the origin is the set of points M in the plane located at constant distance from the origin.
  - The constant distance is the radius r of the circle.
  - The origin is the centre of the circle.



$M(x, y) \in \mathscr{C} \Leftrightarrow d(0, M) = r$

- The standard form of the equation of a circle of radius r centred at $0(0, 0)$ is:

$$x^2 + y^2 = r^2$$

1. Find the equation of the circle centred at the origin with radius

a) $r = 2$ $x^2 + y^2 = 4$ **b)** $r = \sqrt{3}$ $x^2 + y^2 = 3$

2. Find the equation of the circle centred at the origin passing through $A(-2, 3)$. $x^2 + y^2 = 13$

3. Consider the circle with equation: $x^2 + y^2 = 5$. Indicate if the following points belong to the circle.

a) $A(-2, 1)$ *Yes* **b)** $B(1, -2)$ *Yes* **c)** $C(-2, 2)$ *No*

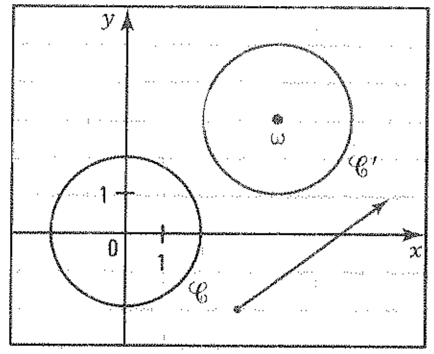
4. Consider the circle with equation: $x^2 + y^2 = 25$. Find the points $M(x, y)$ of the circle that have

a) an x-coordinate equal to 4. $M_1(4, 3)$ *and* $M_2(4, -3)$

b) a y-coordinate equal to -2. $M_1\left(-\sqrt{21}, -2\right)$ *and* $M_2\left(\sqrt{21}, 2\right)$

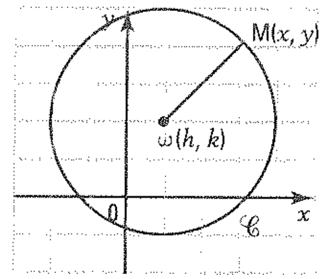
ACTIVITY 2 Circle not centred at the origin

a) The equation of the circle $\mathscr{C}$ on the right, centred at O(0, 0) with radius 2, is: $x^2 + y^2 = 4$. The image of circle $\mathscr{C}$ under the translation $t: (x, y) \Rightarrow (x + 4, y + 3)$ is circle $\mathscr{C}'$.



1. What are the coordinate of the centre ω of circle $\mathscr{C}'$?
   ω(4, 3)
2. What is the radius of circle $\mathscr{C}'$? ***2***
3. What is the equation of circle $\mathscr{C}'$? ***$(x - 4)^2 + (y - 3)^2 = 4$***

b) A circle $\mathscr{C}$ of radius r centred at $\omega(h, k)$ is represented on the right.



1. What is the property which characterizes every point M on this circle?
   Every point M on this circle is located at a distance r from the centre. $M \in \mathscr{C} \Leftrightarrow d(\omega, M) = r$.
2. Justify the steps allowing us to find the equation of the circle in the standard form.

	Steps	Justifications
1.	$M(x, y) \in \mathscr{C} \Leftrightarrow d(\omega, M) = r$	***Definition of the circle.***
2.	$\Leftrightarrow \sqrt{(x-h)^2 + (y-k)^2} = r$	***Formula for the distance between 2 points.***
3.	$\Leftrightarrow (x - h)^2 + (y - k)^2 = r^2$	***When a and b are positive, we have the equivalence: $a = b \Leftrightarrow a^2 = b^2$.***

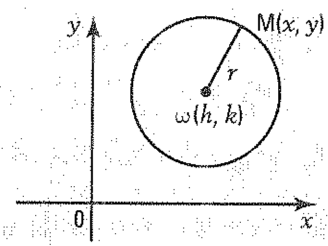
EQUATION OF A CIRCLE: STANDARD FORM

- The set of points M in the plane located at the same distance r from the centre ω, written $\mathscr{C}(\omega, r)$, is called **circle** $\mathscr{C}$ of radius r centred at ω.

$$M \in \mathscr{C}(\omega, r) \Leftrightarrow d(\omega, M) = r$$

- The **standard form** of the equation of a circle of radius r centred at $\omega(h, k)$ is:

$$(x - h)^2 + (y - k)^2 = r^2$$



Ex.: The equation, in standard form, of the circle of radius $r = 5$ centred at $\omega(-3, 2)$ is:
$(x + 3)^2 + (y - 2)^2 = 25$.

5. Consider the circle $\mathscr{C}$ of radius 3 units centred at O(0, 0) and the translation $t: (x, y) \rightarrow (x - 1, y + 2)$.

a) Find the equation of circle $\mathscr{C}$. $x^2 + y^2 = 9$

b) We draw circle $\mathscr{C}'$, image of circle $\mathscr{C}$, under translation t.

1. Determine the coordinates of the centre of circle $\mathscr{C}'$ and its radius.
   Centre (−1, 2); radius: 3
2. Find the equation of circle $\mathscr{C}'$. $(x + 1)^2 + (y - 2)^2 = 9$

6. Determine the centre ω and the radius r of the following circles.

a) $(x - 3)^2 + (y - 4)^2 = 16$ $\omega(3, 4)$; $r = 4$

b) $(x + 2)^2 + (y - 1)^2 = 9$ $\omega(-2, 1)$; $r = 3$

c) $(x + 3)^2 + (y + 1)^2 = 17$ $\omega(-3, -1)$; $r = \sqrt{17}$

d) $\frac{(x+2)^2}{3} + \frac{(y-7)^2}{3} = 12$ $\omega(-2, 7)$; $r = 6$

7. Find the equation of the circle, in the standard form, knowing the centre ω and a point M on the circle.

a) ω(1, 3) and M(1, 7) $(x - 1)^2 + (y - 3)^2 = 16$

b) ω(−4, 5) and M(2, −3) $(x + 4)^2 + (y - 5)^2 = 100$

8. The points A(−1, 2) and B(−3, −4) are the endpoints of a diameter AB of a circle. Find the equation, in the standard form, of this circle.
$(x + 2)^2 + (y + 1)^2 = 10$

9. The equation of a circle centred at the origin is $x^2 + y^2 = 16$.
Describe the translation which associates, with this circle, the circle of equation

a) $(x - 1)^2 + (y + 4)^2 = 16$: $(x, y) \rightarrow (x + 1, y - 4)$

b) $x^2 + (y - 3)^2 = 16$: $(x, y) \rightarrow (x, y + 3)$

c) $(x + 2)^2 + y^2 = 16$: $(x, y) \rightarrow (x - 2, y)$

d) $(x + 1)^2 + (y + 3)^2 = 16$: $(x, y) \rightarrow (x - 1, y - 3)$

ACTIVITY 3 Equation of a circle: general form

a) Consider the circle of radius $r = 4$ centred at $\omega(-1, 2)$.

1. Find the equation of the circle in the standard form. $(x + 1)^2 + (y - 2)^2 = 16$
2. Expand the standard equation in order to write the equation in the form $x^2 + y^2 + ax + by + c = 0$, called **general form** of the equation of a circle.
   $(x + 1)^2 + (y - 2)^2 = 16 \Leftrightarrow x^2 + y^2 + 2x - 4y - 11 = 0$

b) Each of the following expressions is a perfect square trinomial. Complete it and then factor it.

1. $x^2 + 2x + 1 = (x + 1)^2$
2. $x^2 - 6x + 9 = (x - 3)^2$
3. $y^2 + 4y + 4 = (y + 2)^2$
4. $y^2 - 8y + 16 = (y - 4)^2$
5. $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$
6. $y^2 - 5y + \frac{25}{4} = \left(y - \frac{5}{2}\right)^2$

c) The general form of the equation of a circle is: $x^2 + y^2 - 6x + 4y - 12 = 0$.

1. Justify the steps allowing us to write the equation in the standard form.

	Steps	Justifications
1.	$x^2 + y^2 - 6x + 4y - 12 = 0$	- *General equation of the circle.*
2.	$x^2 - 6x + \dots + y^2 + 4y + \dots = 12$	- *We add 12 to each side.*
3.	$x^2 - 6x + 9 + y^2 + 4y + 4 = 12 + 9 + 4$	- *We complete in order to obtain perfect square trinomials.*
4.	$(x - 3)^2 + (y + 2)^2 = 25$	- *We factor the perfect square trinomials to obtain the standard form.*

2. Identify the centre and the radius of the circle. *Centre: (3, –2); radius: 5.*

EQUATION OF A CIRCLE – GENERAL FORM

- Expanding the standard form of the equation of a circle $(x - h)^2 + (y - k)^2 = r^2$, we obtain the **general form**

$$x^2 + y^2 + ax + by + c = 0$$

Ex.: $(x + 3)^2 + (y - 2)^2 = 25$ (Standard form)

$\Leftrightarrow x^2 + 6x + 9 + y^2 - 4y + 4 = 25$

$\Leftrightarrow x^2 + y^2 + 6x - 4y - 12 = 0$ (General form)

- The curve with equation: $x^2 + y^2 + ax + by + c = 0$ is represented by a circle in the Cartesian plane if and only if $a^2 + b^2 > 4c$.
- From the general form of the equation of a circle, we can find the standard form of the circle equation (See activity 3c)).

10. In each of the following cases, find the equation of circle $\mathscr{C}$ in the standard form and then in the general form.

a) Centre (–1, 3); radius 2.
1. $(x + 1)^2 + (y - 3)^2 = 4$
2. $x^2 + y^2 + 2x - 6y + 6 = 0$

b) Centre (0, –3); radius 2.
1. $x^2 + (y + 3)^2 = 4$
2. $x^2 + y^2 + 6y + 5 = 0$

c) Centre (2, –1); M(–1, 3) $\in \mathscr{C}$.
1. $(x - 2)^2 + (y + 1)^2 = 25$
2. $x^2 + y^2 - 4x + 2y - 20 = 0$

d) $\overline{AB}$ is a diameter, A(–2, 1) and B(4, –3).
1. $(x - 1)^2 + (y + 1)^2 = 13$
2. $x^2 + y^2 - 2x + 2y - 11 = 0$

11. For each of the following circles,
1. write the equation of the circle in the standard form.
2. find the centre and the radius of the circle.

a) $x^2 + y^2 - 2x + 4y - 4 = 0$

$(x - 1)^2 + (y + 2)^2 = 9$

$\omega(1, -2)$; $r = 3$

b) $x^2 + y^2 + 8x + 4y + 19 = 0$

$(x + 4)^2 + (y + 2)^2 = 1$

$\omega(-4, -2)$; $r = 1$

c) $x^2 + y^2 - 4x + 6y - 4 = 0$

$(x - 2)^2 + (y + 3)^2 = 17$

$\omega(2, -3)$; $r = \sqrt{17}$

d) $x^2 + y^2 - x + 3y - 1.5 = 0$

$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 4$

$\omega\left(\frac{1}{2}, \frac{-3}{2}\right)^2$; $r = 2$

12. Explain why each of the following equations is not that of a circle.

a) $x^2 + y^2 - 4x + 6y + 14 = 0$ ***$(x - 2)^2 + (y + 3)^2 = -1$. The sum of 2 squares cannot be negative.***

b) $x^2 - y^2 - 2x - 4y - 19 = 0$ ***The coefficient of y^2 cannot be negative.***

13. Determine, if they exist, the intersection points of circle $\mathscr{C}$ with line l.

a) $\mathscr{C}: (x - 1)^2 + (y + 2)^2 = 9$
$l: y = -x + 2$
(1, 1) and (4, −2)

b) $\mathscr{C}: (x + 1)^2 + (y - 2)^2 = 2$
$l: x - y + 1 = 0$
(0, 1)

c) $\mathscr{C}: (x + 2)^2 + (y + 1)^2 = 1$
$l: x - y - 1 = 0$
No intersection point.

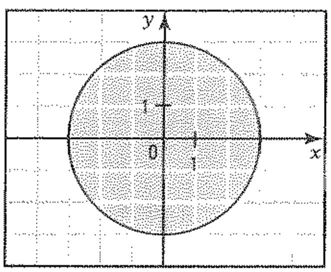
d) $\mathscr{C}: x^2 + y^2 - 2x + 4y + 1 = 0$
$l: x - y - 1 = 0$
(1, 0) and (−1, −2)

e) $\mathscr{C}: x^2 + y^2 = 9$
$l: x + y = 5$
No intersection point.

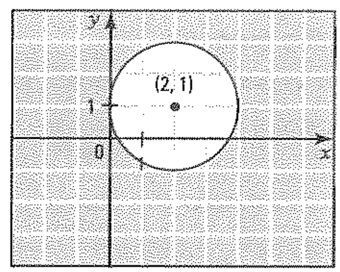
f) $\mathscr{C}: x^2 + y^2 - 2x + 4y - 20 = 0$
$l: 3x + 4y - 20 = 0$
(4, 2)

14. Represent the solution set of the following inequalities in the Cartesian plane.

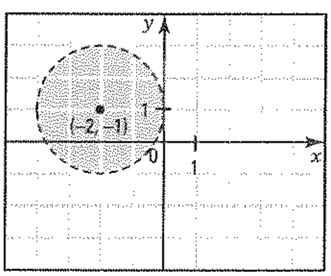
a) $x^2 + y^2 \leqslant 9$



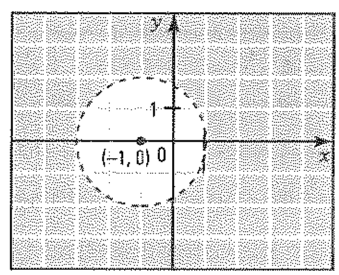
b) $(x - 2)^2 + (y - 1)^2 \geqslant 4$



c) $x^2 + y^2 + 4x + 2y + 1 < 0$

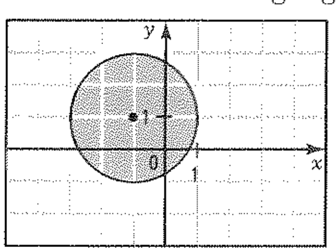


d) $x^2 + y^2 + 2x - 3 > 0$



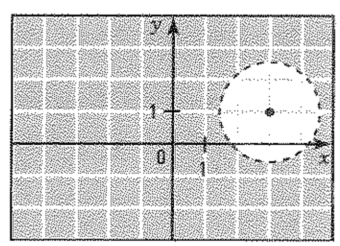
15. For each of the following regions, determine the inequality that defines it.

a)



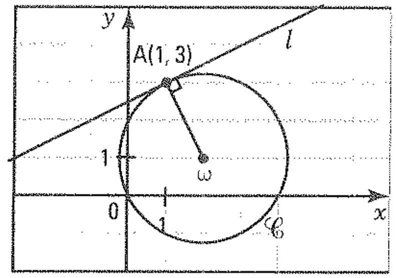
$(x + 1)^2 + (y - 1)^2 \leqslant 4$

b)



$(x - 3)^2 + (y - 1)^2 > \frac{9}{4}$

ACTIVITY 4 Line tangent to a circle



Consider on the right the circle 𝒞 with equation: $(x - 2)^2 + (y - 1)^2 = 5$ and point A(1, 3) on this circle.

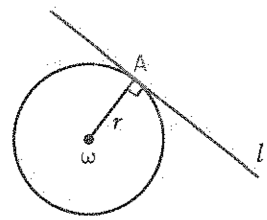
a) Find the coordinates of the centre ω of this circle. *ω(2, 1)*

b) Draw the line l passing through point A and perpendicular to the radius ωA.

This line l intersects circle 𝒞 in only one point. Line l is called **tangent** to the circle at point A.

c) Find the equation of line l. $y = \frac{1}{2}x + \frac{5}{2}$

LINE TANGENT TO A CIRCLE



- A line is **tangent** to a circle if it intersects the circle in only one point called **point of tangency**.

  Ex.: Given the circle on the right,
  - line l is tangent to the circle at point A.
  - A is the point of tangency.

- **Properties of the tangent:**
  - The tangent is perpendicular to the radius at the point of tangency.

    $$l \perp \overline{\omega A}$$

  - The distance between the centre ω and the tangent l is equal to the radius r.

    $$d(\omega, l) = r$$

  - Given a point A on a circle, there exists **only one line** tangent to the circle at point A.

16. Consider the circle with equation: $(x + 1)^2 + (y - 2)^2 = 25$.

a) Verify that point A(2, 6) is a point on this circle. $(2 + 1)^2 + (6 - 2)^2 = 25$

b) What are the coordinates of the centre ω of the circle? *ω(−1, 2)*

c) Explain the procedure to find the equation of the line l tangent to the circle at point A.

1. We calculate the slope of the radius $\overline{\omega A}$.

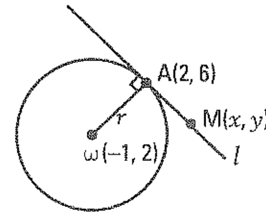
2. We deduce the slope of the tangent l knowing that $\overline{\omega A} \perp l$.

3. We find the equation of the line l passing through point A(2, 6) knowing its slope.

d) Find the equation of the tangent l at point A.

1. $a_{\omega a} = \frac{6 - 2}{2 + 1} = \frac{4}{3}$ *2.* $a_d = \frac{-3}{4}$ *3.* $d: y = -\frac{3}{4}x + \frac{15}{2}$

17. Use the data from exercise 16. Justify the steps allowing us to find the equation of the tangent l at point A using the scalar product.



	Steps	Justifications
1.	$M(x, y) \in d \Leftrightarrow \overrightarrow{AM} \perp \overrightarrow{A\omega}$	*Property of the tangent.*
2.	$\Leftrightarrow \overrightarrow{AM} \cdot \overrightarrow{A\omega} = 0$	*Property of the scalar product.*
3.	$\Leftrightarrow (x-2, y-6) \cdot (-3, -4) = 0$	*Calculating the vector components.*
4.	$\Leftrightarrow -3(x-2) - 4(y-6) = 0$	*Calculating the scalar product.*
5.	$\Leftrightarrow -3x - 4y + 30 = 0$	*Equation of line l (general form).*
6.	$\Leftrightarrow y = -\frac{3}{4}x + \frac{15}{2}$	*Equation of line l (functional form).*

18. Find the equation of the tangent to the circle with equation: $x^2 + y^2 - 2x + 4y - 35 = 0$ at point A(3, 4).

$\mathscr{C}: (x-1)^2 + (y+2)^2 = 40;\ \omega(1, -2);\ a_{\omega A} = 3;\ l: y = -\frac{1}{3}x + 5$

19. Find the equation of the circle with centre ω(–2, 3) if the circle is tangent to

a) the x-axis. $(x+2)^2 + (y-3)^2 = 9$ **b)** the y-axis. $(x+2)^2 + (y-3)^2 = 4$

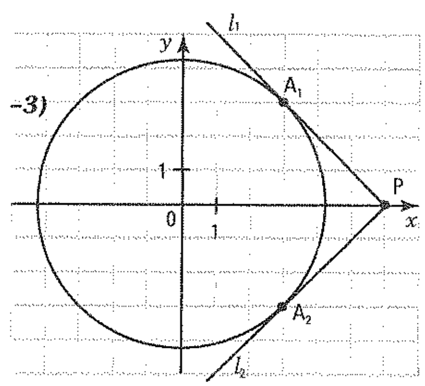
20. Consider the line l: $2x + y - 4 = 0$ and a point ω(1, –3).

a) Find the equation of the circle $\mathscr{C}$ centred at ω and tangent to line l.

Radius $= d(\omega, l) = \sqrt{5}$; $(x-1)^2 + (y+3)^2 = 5$

b) Find the coordinates of the point of tangency A. A(3, –2)

21. Consider the circle $\mathscr{C}$ with equation: $x^2 + y^2 = 18$.



a) Find the coordinates of points A_1 and A_2 on the circle which have an x-coordinate equal to 3. $A_1(3, 3)$ and $A_2(3, -3)$

b) Find the equation of the lines l_1 and l_2 tangent to circle $\mathscr{C}$ at points A_1 and A_2 respectively.

$l_1: y = -x + 6$ and $l_2: y = x - 6$

c) Show that lines l_1 and l_2 meet at a point P located on the x-axis.

The system $\begin{cases} y = -x + 6 \\ y = x - 6 \end{cases}$ *has solution* P(6, 0)

d) Show that the quadrilateral $0A_1PA_2$ is a square.

$m\overline{0A_1} = m\overline{A_1P} = m\overline{PA_2} = m\overline{0A_2} = \sqrt{18} \Rightarrow 0A_1PA_2$ *is a rhombus.* $\angle 0A_1P$ *is a right angle (Property of the tangent)* $\Rightarrow 0A_1PA_2$ *is a square, since one of the angles of the rhombus is right.*

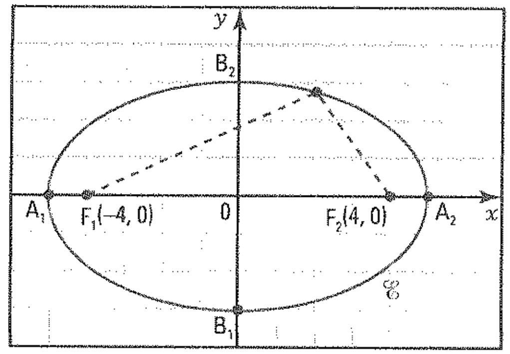
e) Calculate the area of the region bounded by the line segment PA_1, the line segment PA_2 and the arc of circle A_1A_2. *Area of square* $0A_1PA_2 = 18\ u^2$

Area of the circular sector $A_10A_2 = \frac{18\pi}{4} u^2$ *(one quarter of the disk). Requested area* $= \left(18 - \frac{9\pi}{2}\right) u^2$.

7.3 Ellipse

Activity 1 Ellipse centred at the origin

Consider two points $F_1(-4, 0)$ and $F_2(4, 0)$ located on the x-axis.
The set of points M in the plane such that the sum of the distances $d(M, F_1) + d(M, F_2) = 10$ is an **ellipse** with foci at points F_1 and F_2.
The four points $A_1(-5, 0)$, $A_2(5, 0)$, $B_1(0, -3)$ and $B_2(0, 3)$ represent the **vertices** of the ellipse.



a) Verify that

1. $d(A_1, F_1) + d(A_2, F_2) = 10$ ***$d(A_1, F_1) = 1$; $d(A_1, F_2) = 9$***
2. $d(A_2, F_1) + d(A_2, F_2) = 10$ ***$d(A_2, F_1) = 9$; $d(A_2, F_2) = 1$***
3. $d(B_1, F_1) + d(B_1, F_2) = 10$ ***$d(B_1, F_1) = 5$; $d(B_1, F_2) = 5$***
4. $d(B_2, F_1) + d(B_2, F_2) = 10$ ***$d(B_2, F_1) = 5$; $d(B_2, F_2) = 5$***

b) Justify the steps allowing us to find the equation of this ellipse $\mathscr{E}$.

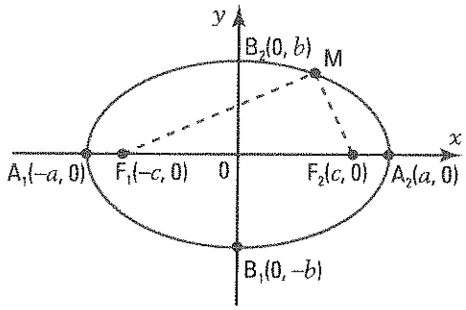
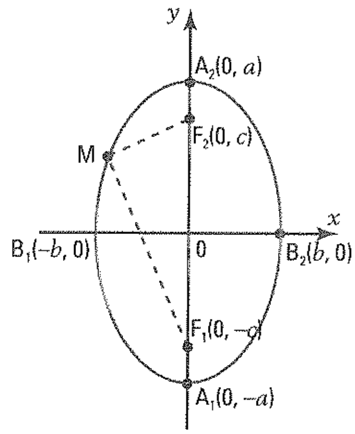
	Steps	Justifications
1.	$M(x, y) \in \mathscr{E} \Leftrightarrow d(M_1F_2) + d(M_1F_2) = 10$	*Definition of ellipse ℰ.*
2.	$\Leftrightarrow \sqrt{(x+4)^2 + y^2} + \sqrt{(x-4)^2 + y^2} = 10$	*Formula for the distance between 2 points.*
3.	$\Leftrightarrow \sqrt{(x+4)^2 + y^2} = 10 - \sqrt{(x-4)^2 + y^2}$	*Property of the equality relation.*
4.	$\Leftrightarrow (x+4)^2 + y^2 = 100 + (x-4)^2 + y^2 - 20\sqrt{(x-4)^2 + y^2}$	*Squaring both sides.*
5.	$\Leftrightarrow x^2 + 8x + 16 + y^2 = 100 + x^2 - 8x + 16 + y^2 - 20\sqrt{(x-4)^2 + y^2}$	*Expansion of the squares.*
6.	$\Leftrightarrow 16x - 100 = -20\sqrt{(x-4)^2 + y^2}$	*Reduction.*
7.	$\Leftrightarrow (16x - 100)^2 = 400[(x-4)^2 + y^2]$	*Squaring both sides.*
8.	$\Leftrightarrow 256x^2 - 3200x + 10\,000 = 400(x^2 - 8x + 16 + y^2)$	*Expansion of the squares.*
9.	$\Leftrightarrow 256x^2 - 3200x + 10\,000 = 400x^2 - 3200x + 6400 + 400y^2$	*Distributivity.*
10.	$\Leftrightarrow 144x^2 + 400y^2 = 3600$	*Reduction.*
11.	$\Leftrightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$	*We divide both sides by 3600.*

c) The equation of the ellipse in b) is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > 0, b > 0)$ called **standard form.**

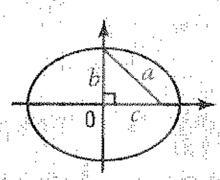
1. Identify parameters a and b. ***$a = 5$ and $b = 3$***
2. Verify that the major axis A_1A_2 containing the foci has a length of $2a$. ***$a = 5$; $d(A_1A_2) = 10$***
3. Verify that the minor axis B_1B_2 has a length of $2b$. ***$b = 3$; $d(B_1B_2) = 6$***
4. If $2c$ is the distance between the foci, identify c and verify that $b^2 + c^2 = a^2$.
   $c = 4$; $b^2 + c^2 = 3^2 + 4^2 = 5^2$

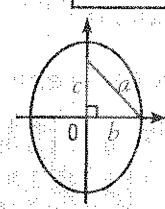
ELLIPSE CENTRED AT THE ORIGIN

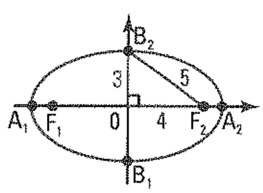
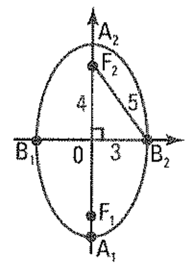
- The ellipse with **foci** F_1 and F_2 is the set of all points M of the plane for which the sum of the distances to the foci is **constant**.
- We distinguish two cases according to the position of the **major axis** containing the foci.

Horizontal major axis	Vertical major axis
	
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$

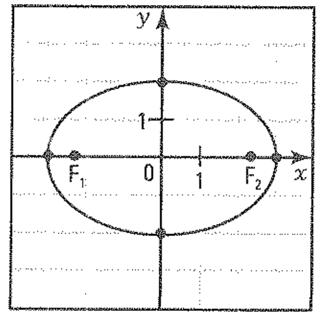
- For every point M on the ellipse we have: $d(M, F_1) + d(M, F_2) = 2a$
- The four **vertices** of the ellipse are: A_1, A_2, B_1 and B_2.
- The **major axis** is the line segment A_1A_2 with length $2a$. $(a > 0)$
  The foci F_1 and F_2 are located on the major axis.
- The **minor axis** is the line segment B_1B_2 with length $2b$. $(b > 0)$
- The **focal distance** is the length $2c$ between the foci F_1 and F_2. $(c > 0)$
- The axes of the ellipse are axes of symmetry.
- The centre 0 of the ellipse is a centre of symmetry.
- The parameters a, b and c of the ellipse verify Pythagoras' relation: $a^2 = b^2 + c^2$





Ex.: $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$	Ex.: $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$
We have: $a = 5$, $b = 3$, $c = 4$ $A_1(-5, 0)$, $A_2(0, 5)$ $B_1(0, -3)$, $B_2(0, 3)$ $F_1(-4, 0)$, $F_2(4, 0)$ 	We have: $a = 5$, $b = 3$, $c = 4$ $A_1(0, -5)$, $A_2(0, 5)$ $B_1(-3, 0)$, $B_2(3, 0)$ $F_1(0, -4)$, $F_2(0, 4)$ 

1. Consider the ellipse with equation: $\frac{x^2}{9}+\frac{y^2}{4}=1$.



a) Draw the ellipse in the Cartesian plane on the right.

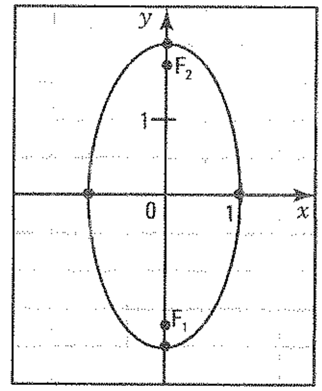
b) Is the major axis horizontal or vertical? ***Horizontal***

c) Find the coordinates of the vertices. ***(−3, 0), (3, 0), (0, −2), (0, 2)***

d) Find the coordinates of the foci F_1 and F_2 and locate the foci in the Cartesian plane. ***$F_1(-\sqrt{5}, 0)$, $F_2(\sqrt{5}, 0)$***

e) If M is a point on the ellipse, determine $d(M, F_1) + d(M, F_2)$. ***6***

2. Consider the ellipse with equation: $\frac{x^2}{1}+\frac{y^2}{4}=1$.



a) Draw the ellipse in the Cartesian plane on the right.

b) Is the major axis horizontal or vertical? ***Vertical***

c) Find the coordinates of the vertices. ***(−1, 0), (1, 0), (0, −2), (0, 2)***

d) Find the coordinates of the foci F_1 and F_2 and locate the foci in the Cartesian plane.
$F_1(-\sqrt{3}, 0)$, $F_2(0, \sqrt{3})$

e) If M is a point on the ellipse, determine $d(M, F_1) + d(M, F_2)$. ***4***

3. a) Explain the steps allowing us to write the standard form of the equation of an ellipse centred at the origin in the general form $ax^2 + by^2 + c = 0$.

$\frac{x^2}{9}+\frac{y^2}{4}=1$ (standard form)

1. $36\left(\frac{x^2}{9}+\frac{y^2}{4}\right)=36$ ***We multiply both sides by the common denominator.***
2. $4x^2 + 9y^2 = 36$ ***We apply distributivity.***
3. $4x^2 + 9y^2 - 36 = 0$ ***We subtract 36 from both sides.***

b) Write the equations of the following ellipses in the general form.

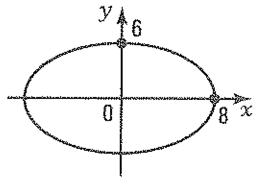
1. $\frac{x^2}{4}+\frac{y^2}{5}=1$ ***$5x^2 + 4y^2 - 20 = 0$***
2. $x^2+\frac{y^2}{9}=1$ ***$9x^2 + y^2 - 9 = 0$***
3. $\frac{x^2}{\frac{4}{9}}+\frac{y^2}{16}=1$ ***$36x^2 + y^2 - 16 = 0$***

4. Complete the following table.

Equation of the ellipse	Position of the major axis	Coordinates of the vertices	Coordinates of the foci	Length of the major axis	Length of the minor axis
$\frac{x^2}{169}+\frac{y^2}{25}=1$	*Horizontal*	*(−13, 0), (13, 0), (0, −5), (0, 5)*	*(−12, 0), (12, 0)*	*26*	*10*
$\frac{x^2}{36}+\frac{y^2}{100}=1$	*Vertical*	*(−6, 0), (6, 0), (0, −10), (0, 10)*	*(0, −8), (0, 8)*	*20*	*12*
$x^2 + 4y^2 - 4 = 0$	*Horizontal*	*(−2, 0), (2, 0), (0, −1), (0, 1)*	*$(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$*	*4*	*2*
$25x^2 + 4y^2 - 100 = 0$	*Vertical*	*(−2, 0), (2, 0), (0, −5), (0, 5)*	*$(0, -\sqrt{21})$, $(0, \sqrt{21})$*	*10*	*4*

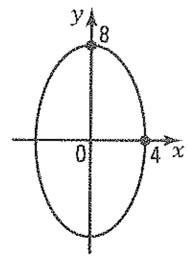
5. Determine the equation of each of the following ellipses.

a)



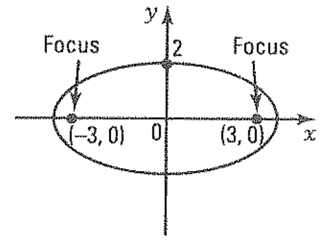
$\frac{x^2}{64} + \frac{y^2}{36} = 1$

b)



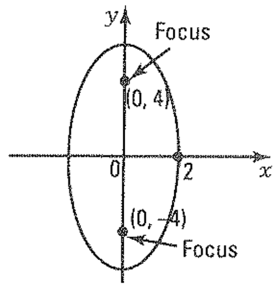
$\frac{x^2}{16} + \frac{y^2}{64} = 1$

c)



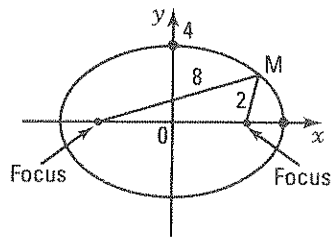
$\frac{x^2}{13} + \frac{y^2}{4} = 1$

d)



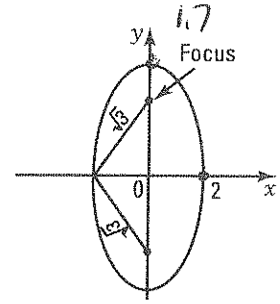
$\frac{x^2}{4} + \frac{y^2}{20} = 1$

e)



$\frac{x^2}{25} + \frac{y^2}{16} = 1$

f)



$\frac{x^2}{4} + \frac{y^2}{3} = 1$

6. Determine the coordinates of the foci of an ellipse centred at the origin whose vertices are the points

a) (6, 0) and (0, 4) ***$(-\sqrt{20}, 0)$ and $(0, \sqrt{20})$***
b) (6, 0) and (0, 7) ***$(0, -\sqrt{13})$ and $(0, \sqrt{13})$***
c) (−10, 0) and (0, −6) ***(−8, 0) and (8, 0)***
d) (−4, 0) and (0, −6) ***$(0, -\sqrt{20})$ and $(0, \sqrt{20})$***

7. In each of the following cases, find the equation of the ellipse in the standard form.

a) The major axis of length 20 units is horizontal and the minor axis measures 12 units.
$\frac{x^2}{100} + \frac{y^2}{36} = 1$

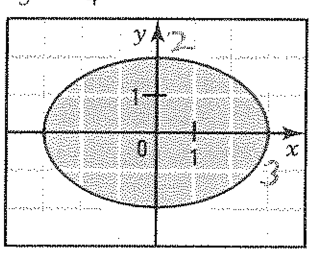
b) The major axis of length 16 units is vertical and the minor axis measures 10 units.
$\frac{x^2}{25} + \frac{y^2}{64} = 1$

c) The major axis of length 10 units is horizontal and the focal distance measures 6 units.
$\frac{x^2}{25} + \frac{y^2}{16} = 1$

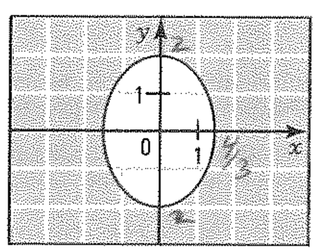
d) The major axis of length 20 units is vertical and the focal distance measures 16 units.
$\frac{x^2}{36} + \frac{y^2}{100} = 1$

8. Represent the following inequalities in the Cartesian plane.

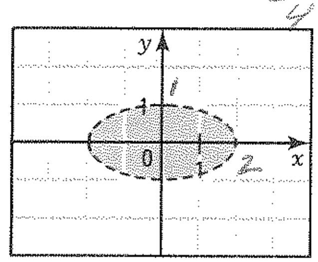
a) $\frac{x^2}{9} + \frac{y^2}{4} \leq 1$



b) $9x^2 + 4y^2 - 16 \geq 0$

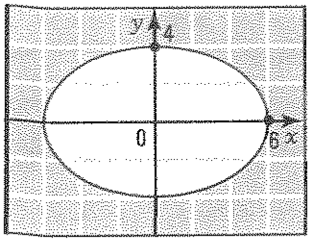


c) $x^2 + 4y^2 - 4 < 0$



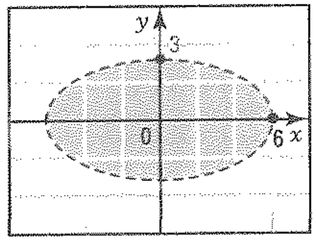
9. Describe each of the following regions using an inequality.

a)



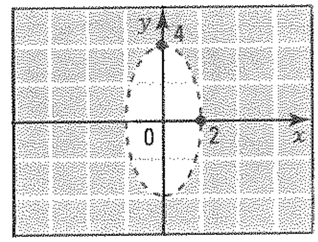
$\frac{x^2}{36} + \frac{y^2}{16} \geq 1$

b)



$\frac{x^2}{36} + \frac{y^2}{9} < 1$

c)

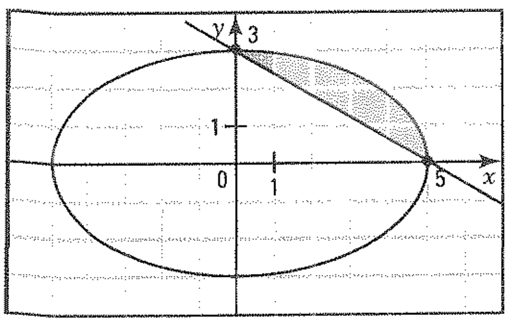


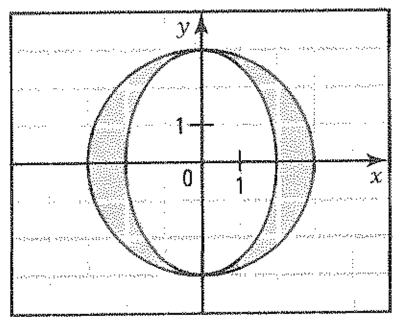
$\frac{x^2}{4} + \frac{y^2}{16} > 1$

10. Represent the following systems in the Cartesian plane.

a) $\begin{cases} 9x^2 + 25y^2 - 225 \leq 0 \\ 3x + 5y - 15 \geq 0 \end{cases}$

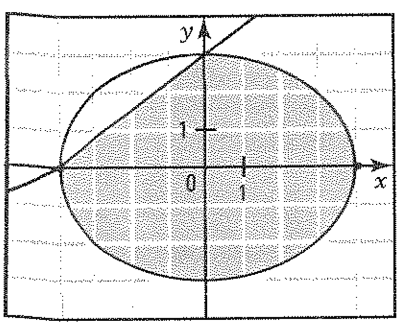
b) $\begin{cases} x^2 + y^2 - 9 \leq 0 \\ 9x^2 + 4y^2 - 36 \geq 0 \end{cases}$





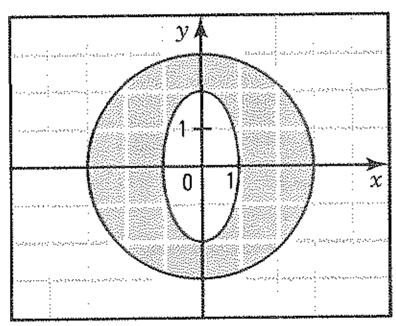
11. Describe each of the following shaded regions using a system of inequalities.

a)



$\begin{cases} \frac{x^2}{16} + \frac{y^2}{9} \leq 1 \\ 3x - 4y + 12 \geq 0 \end{cases}$

b)



$\begin{cases} x^2 + y^2 \leq 9 \\ x^2 + \frac{y^2}{4} \geq 1 \end{cases}$

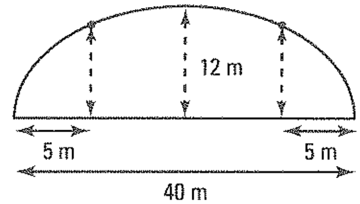
12. Determine the coordinates of the intersection points of the ellipse with equation $x^2 + 4y^2 - 25 = 0$ and the line with equation $x + 2y - 7 = 0$.

Points (3, 2) and $\left(4, \frac{3}{2}\right)$.

13. An art exhibit is presented in a room whose ceiling has the shape of a half-ellipse as shown on the figure on the right.

Spotlights are installed on the ceiling at a horizontal distance of 5 m from the edges of the ceiling.

What is the height of the spotlights, as measured from the floor?

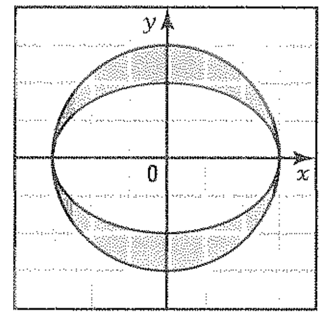


Ellipse: $\frac{x^2}{400} + \frac{y^2}{144} = 1$; when x = 15, y = $\pm\sqrt{63}$. The spotlights are at height 7.94 m.

14. The circle with equation: $x^2 + y^2 = 9$ and the ellipse with equation: $4x^2 + 9y^2 - 36 = 0$ are represented on the right. Calculate the area of the shaded region.

Note: The area of an ellipse with major axis measuring $2a$ and minor axis measuring $2b$ is equal to πab.

a = 3; b = 2; Area of shaded region = 9π − 6π = 3π u²



15. A race track has an elliptical shape. The length of the major axis is 80 m and the focal distance is 64 m.

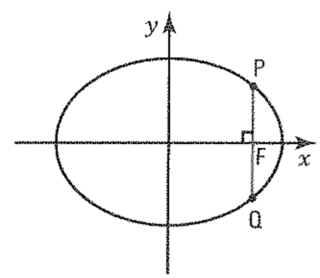
What is the length of the minor axis? ***48 m***

16. The measure of the chord perpendicular to the major axis and passing through one of the foci of an ellipse is called **latus rectum**.

Calculate the measure of the latus rectum of an ellipse defined by the equation $9x^2 + 25y^2 - 225 = 0$.

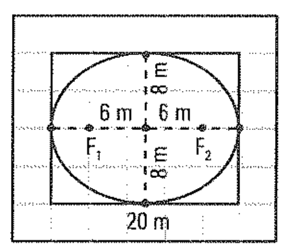
a = 5; b = 3; c = 4; $y_P = \frac{9}{5}$; $y_Q = -\frac{9}{5}$

Measure of the latus rectum = 3.6 u.



17. A rectangular garden has a perimeter of 72 m and an area of 320 m². A landscape architect wants to create a flowerbed shaped as an ellipse in this garden. Using a sketch, give the dimensions of the ellipse and locate the foci so that the flowerbed has maximum area.

Major axis = 20 m; minor axis = 16 m; focal distance = 12 m.

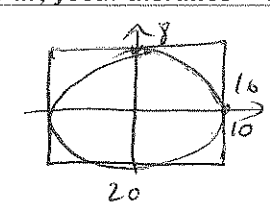


$a + b = 36$

$ab = 320$

$20 \times 16 = 320$

$20 + 16 = 36$



$\frac{x^2}{10^2} + \frac{y^2}{8^2} = 1$

$c^2 = a^2 - b^2$

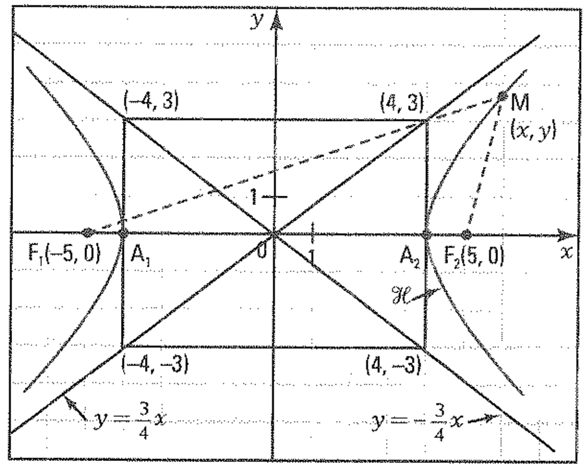
$6^2 = a^2 - 8^2$

$a = 10$

7.4 Hyperbola

ACTIVITY 1 Hyperbola centred at the origin

Consider two points $F_1(-5, 0)$ and $F_2(5, 0)$ located on the x-axis.
The set of points M of the plane such that the absolute value of the difference $|d(M, F_1) - d(M, F_2)| = 8$ is a **hyperbola** whose **foci** are the points F_1 and F_2.
The two points $A_1(-4, 0)$ and $A_2(4, 0)$ represent the **vertices** of the hyperbola.



a) Verify that

1. $|d(A_1, F_1) - d(A_1, F_2)| = 8$
   $d(A_1, F_1) = 1$; $d(A_1, F_2) = 9$; $|1 - 9| = 8$
2. $|d(A_2, F_1) - d(A_2, F_2)| = 8$
   $d(A_2, F_1) = 9$; $d(A_2, F_2) = 1$; $|9 - 1| = 8$

b) Justify the steps allowing us to find the equation of the hyperbola $\mathscr{H}$.

	Étapes	Justifications
1.	$M(x, y) \in \mathscr{H} \Leftrightarrow \lvert d(M, F_1) - d(M, F_2)\rvert = 8$	***Definition of hyperbola $\mathscr{H}$.***
2.	$\Leftrightarrow \left\lvert\sqrt{(x+5)^2 + y^2} - \sqrt{(x-5)^2 + y^2}\right\rvert = 8$	***Formula for the distance between 2 points.***
3.	$\Leftrightarrow \sqrt{(x+5)^2 + y^2} - \sqrt{(x-5)^2 + y^2} = \pm 8$	***Two opposite numbers have the same absolute value.***
4.	$\Leftrightarrow \sqrt{(x+5)^2 + y^2} = \sqrt{(x-5)^2 + y^2} \pm 8$	***Property of the equality relation.***
5.	$\Leftrightarrow (x+5)^2 + y^2 = (x-5)^2 + y^2 + 64 \pm 16\sqrt{(x-5)^2 + y^2}$	***Squaring both sides.***
6.	$\Leftrightarrow x^2 + 10x + 25 + y^2 = x^2 - 10x + 25 + y^2 + 64 \pm 16\sqrt{(x-5)^2 + y^2}$	***Expansion of the squares.***
7.	$\Leftrightarrow 20x - 64 = \pm 16\sqrt{(x-5)^2 + y^2}$	***Reduction.***
8.	$\Leftrightarrow (20x - 64)^2 = 256\left[(x-5)^2 + y^2\right]$	***Squaring both sides.***
9.	$\Leftrightarrow 400x^2 - 2560x + 4096 = 256x^2 - 2560x + 6400 + 256y^2$	***Expansion of the squares.***
10.	$\Leftrightarrow 144x^2 - 256y^2 = 2304$	***Reduction.***
11.	$\Leftrightarrow \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$	***We divide both sides by 2304.***

The equation of the hyperbola obtained in b) is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ called **standard form.**

c) 1. Identify the parameters a and b. ***$a = 4$, $b = 3$***

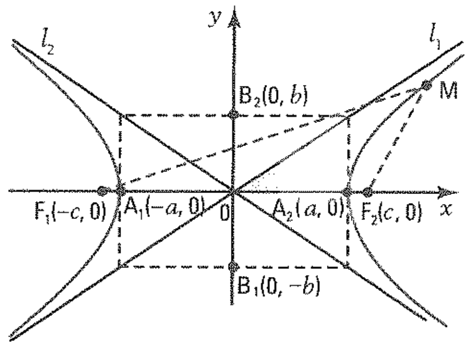
2. If the focal distance (distance between the foci) is $2c$, identify c and verify that $a^2 + b^2 = c^2$. ***$c = 5$; $a^2 = 16$, $b^2 = 9$ and $a^2 + b^2 = c^2$.***

3. Draw the rectangle having vertices: (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$ then verify that the diagonals of this rectangle are supported by the asymptotes of the hyperbola.

HYPERBOLA CENTRED AT THE ORIGIN

- The hyperbola with foci F_1 and F_2 is the set of all points M of the plane such that the absolute value of the difference of the distances to the foci is constant.
- We distinguish two cases according to the position of the **transverse axis** containing the foci.

Horizontal transverse axis

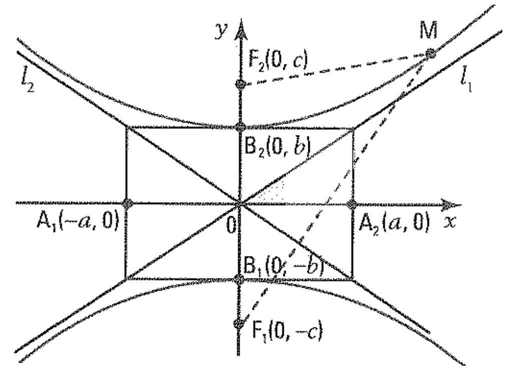


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- The vertices are: $A_1(-a, 0)$ and $A_2(a, 0)$.
- The foci are: $F_1(-c, 0)$ and $F_2(c, 0)$.
- The transverse axis A_1A_2 has length $2a$. $(a > 0)$
- The conjugate axis B_1B_2 has length $2b$. $(b > 0)$
- For every point on the hyperbola, we have:

$$|d(M, F_1) - d(M, F_2)| = 2a$$

Vertical transverse axis



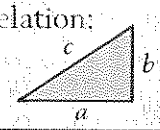
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

- The vertices are: $B_1(0, -b)$ and $B_2(0, b)$.
- The foci are: 0 $F_1(0, -c)$ and $F_2(0, c)$.
- The transverse axis B_1B_2 has length $2b$. $(b > 0)$
- The conjugate axis A_1A_2 has length $2a$. $(a > 0)$
- For every point on the hyperbola, we have:

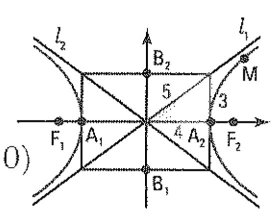
$$|d(M, F_1) - d(M, F_2)| = 2b$$

- The **focal distance** is the distance $2c$ between the foci $(c > 0)$.
- The axes of the hyperbola are axes of symmetry and the origin 0 is a centre of symmetry.
- The hyperbola has two asymptotes passing through the origin: $l_1: y = \frac{b}{a}x$ and $l_2: y = \frac{-b}{a}x$.
- The parameters a, b and c of the hyperbola verify the relation:

$$a^2 + b^2 = c^2$$



Ex.: $\frac{x^2}{16} - \frac{y^2}{9} = 1$



We have:

$a = 4, b = 3, c = 5$

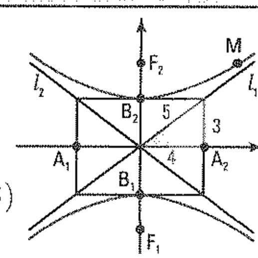
Vertices: $A_1(-4, 0)$; $A_2(4, 0)$

Foci: $F_1(-5, 0)$; $F_2(5, 0)$

For every point M on the hyperbola, we have:
$|d(M, F_1) - d(M, F_2)| = 2a = 8$

Asymptotes: $l_1: y = \frac{3}{4}x$; $l_2: y = -\frac{3}{4}x$.

Ex.: $\frac{x^2}{16} - \frac{y^2}{9} = -1$



We have:

$a = 4, b = 3, c = 5$

Vertices: $B_1(0, -3)$, $B_2(0, 3)$

Foci: $F_1(0, -5)$, $F_2(0, 5)$

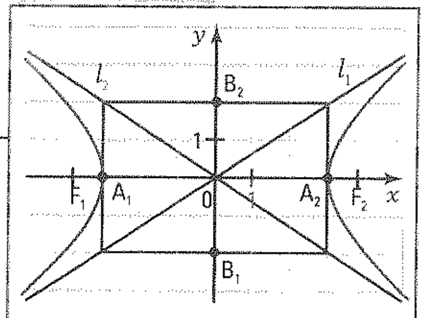
For every point M on the hyperbola, we have:
$|d(M, F_1) - d(M, F_2)| = 2b = 6$

Asymptotes: $l_1: y = \frac{3}{4}x$; $l_2: y = -\frac{3}{4}x$.

DRAWING THE HYPERBOLA

Let us use the following procedure to draw the hyperbola centred at the origin with equation: $\frac{x^2}{9} - \frac{y^2}{4} = 1$

– We identify the position of the transverse axis according to the equation model.	Horizontal transverse axis
– We identify the parameters a and b.	$a = 3; b = 2$
– We deduce the parameter c knowing that $a^2 + b^2 = c^2$.	$c = \sqrt{13}$
– We locate the vertices A_1, A_2 and the foci F_1, F_2 on the transverse axis.	$A_1(-3, 0)$, $A_2(3, 0)$ $F_1(-\sqrt{13}, 0)$, $F_2(\sqrt{13}, 0)$
– We draw the asymptotes l_1 and l_2, supporting the diagonals of the rectangle centred at 0 with vertices (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$.	$l_1 : y = \frac{2}{3}x$ $l_2 : y = \frac{-2}{3}x$
– We draw the hyperbola using a table of values.	



x	3	$\sqrt{13}$	4	5
y	0	$1.\overline{3}$	1.76	$2.\overline{6}$

– We draw one branch of the hyperbola approaching the asymptotes.
– We deduce the other branch by symmetry.

1. Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = -1$.

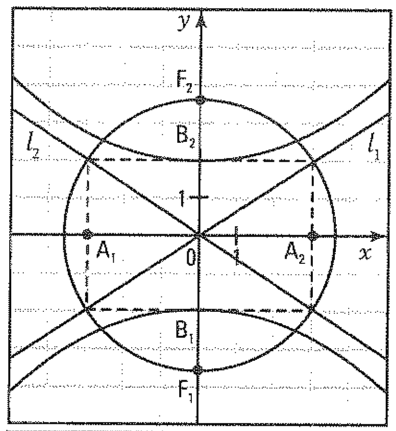
a) What is the position of the transverse axis? ***Vertical***

b) Identify the parameters a, b and c. $a = 3,\ b = 2,\ c = \sqrt{13}$

c) Determine the coordinates of the vertices and the foci.
$B_1(0, -2)$, $B_2(0, 2)$; $F_1(0, -\sqrt{13})$, $F_2(0, \sqrt{13})$

d) Determine
1. the length of the transverse axis. ***4 u***
2. the length of the conjugate axis. ***6 u***
3. the focal distance. $2\sqrt{13}$ ***u*** 7.2

e) Draw the asymptotes of the hyperbola and determine their equation. $l_1 : y = \frac{2}{3}x$; $l_2 : y = \frac{-2}{3}x$



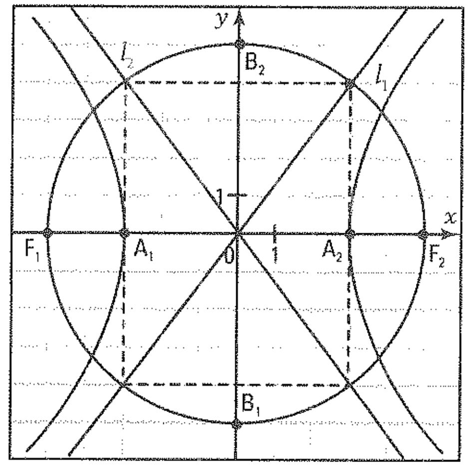
x	0	3.4	4.5	5.2
y	2	3	$\sqrt{13}$	4

f) Draw the hyperbola.

g) If M is a point on the hyperbola, complete: $|d(M, F_1) - d(M, F_2)| =$ $2b = 4$

h) Verify that the circle of radius c centred at 0 passes through the foci of the hyperbola and through the vertices (a, b), $(-a, b)$, $(-a, -b)$ and $(a, -b)$ of the rectangle.

2. Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.



a) What is the position of the transverse axis? ***Horizontal***

b) Identify the parameters a, b and c. ***a = 3, b = 4, c = 5***

c) Determine the coordinates of the vertices and the foci.
$A_1(-3, 0)$, $A_2(3, 0)$; $F_1(-5, 0)$, $F_2(5, 0)$

d) Determine

1. the length of the transverse axis. ***6 u***
2. the length of the conjugate axis. ***8 u***
3. the focal distance. ***10 u***

e) Draw the asymptotes of the hyperbola and determine their equation. ***$l_1: y = \frac{4}{3}x$; $l_2: y = \frac{-4}{3}x$***

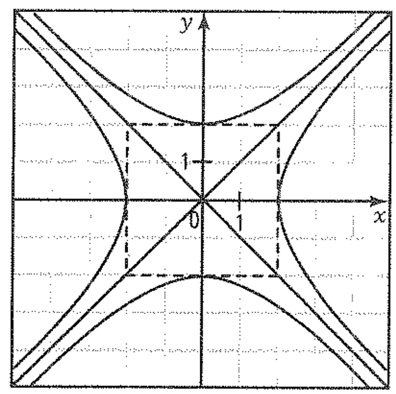
f) Draw the hyperbola.

x	3	4	5
y	***0***	***3.5***	***5.3***

g) If M is a point on the hyperbola, complete: $|d(M, F_1) - d(M, F_2)| =$ ***6***

h) Verify that the circle of radius c centred at 0 passes through the foci of the hyperbola and through the vertices (a, b), $(-a, b)$, $(-a, -b)$ and $(a, -b)$ of the rectangle.

3. When the parameters a and b of the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are equal, the hyperbola is called **equilateral**.



a) 1. Determine the equations of the asymptotes of an equilateral hyperbola. ***y = −x and y = x***

2. Are the asymptotes of an equilateral hyperbola perpendicular? ***Yes***

b) Draw, in the same Cartesian plane,

1. the equilateral hyperbola with equation: $x^2 - y^2 = 4$.
2. the equilateral hyperbola with equation: $x^2 - y^2 = -4$.

4. Write the equation of the following hyperbolas in the general form $ax^2 + by^2 + c = 0$.

a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$9x^2 - 16y^2 - 144 = 0$

b) $\frac{x^2}{36} - \frac{y^2}{64} = -1$

$64x^2 - 36y^2 + 2304 = 0$

$16x^2 - 9y^2 + 576$

c) $\frac{x^2}{\frac{9}{16}} - \frac{y^2}{\frac{25}{16}} = 1$

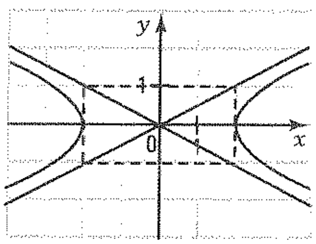
$400x^2 - 144y^2 - 225 = 0$

5. Complete the following table.

Equation of the hyperbola	Position of the transverse axis	Coordinates of the vertices	Coordinates of the foci	Length of transverse axis	Length of conjugate axis	Equations of the asymptotes
$\frac{x^2}{9}-\frac{y^2}{16}=1$	*Horizontal*	*(−3, 0) and (3, 0)*	*(−5, 0) and (5, 0)*	*6*	*8*	$y=\frac{4}{3}x$ $y=\frac{-4}{3}x$
$\frac{x^2}{9}-\frac{y^2}{16}=-1$	*Vertical*	*(0, −4) and (0, 4)*	*(0, −5) and (0, 5)*	*8*	*6*	$y=\frac{4}{3}x$ $y=\frac{-4}{3}x$
$64x^2-36y^2-2304=0$	*Horizontal*	*(−6, 0) and (6, 0)*	*(−10, 0) and (10,0)*	*12*	*16*	$y=\frac{4}{3}x$ $y=\frac{-4}{3}x$
$x^2-2y^2+2=0$	*Vertical*	*(0, −1) and (0, 1)*	$(0,-\sqrt{3})$ *and* $(0,\sqrt{3})$	*2*	$2\sqrt{2}$	$y=\frac{\sqrt{2}}{2}x$ $y=\frac{-\sqrt{2}}{2}x$

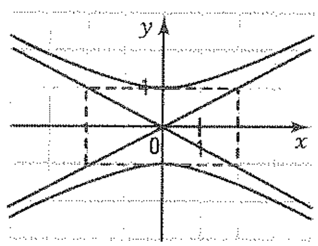
6. For each of the following hyperbolas, determine its equation.

a)



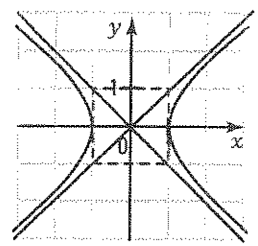
$\frac{x^2}{4}-y^2=1$

b)



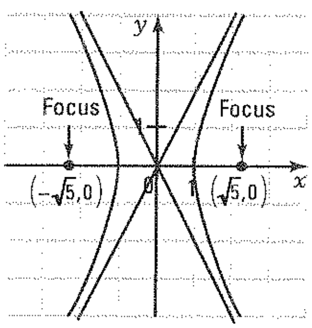
$\frac{x^2}{4}-y^2=-1$

c)



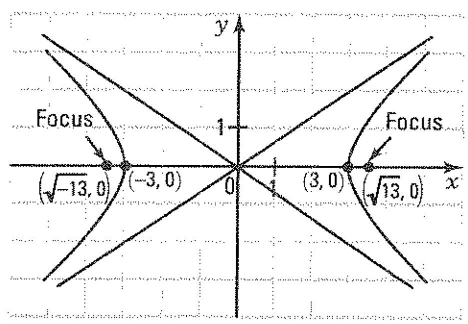
$x^2-y^2=1$

d)



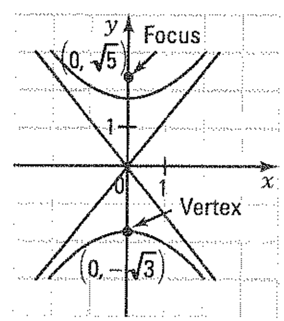
$x^2-\frac{y^2}{4}=1$

e)



$\frac{x^2}{9}-\frac{y^2}{4}=1$

f)



$\frac{x^2}{2}-\frac{y^2}{3}=-1$

7. Determine the coordinates of the foci of a hyperbola centred at the origin in each case.

a) The hyperbola has equation: $\frac{x^2}{3}-\frac{y^2}{2}=-1$. $F_1(0,-\sqrt{13})$; $F_2(0,\sqrt{13})$

b) The hyperbola has equation: $\frac{x^2}{4}-\frac{y^2}{5}=1$. $F_1(-\sqrt{29},0)$; $F_2(\sqrt{29},0)$

c) The focal distance is equal to 12 and the transverse axis is vertical. $F_1(0,-6)$ *and* $F_2(0,6)$

d) (8, 0) is a vertex of the hyperbola and one of the asymptotes has equation: $y=\frac{3}{4}x$.
$F_1(-10,0)$ *and* $F_2(10,0)$

8. Determine the coordinates of the vertices of a hyperbola centred at he origin in each of the following cases.

a) The hyperbola has equation: $\frac{x^2}{4} - \frac{y^2}{16} = -1$. *(0, −4) and (0, 4)*

b) The hyperbola has equation: $\frac{x^2}{5} - \frac{y^2}{4} = 1$. $(-\sqrt{5}, 0)$ *and* $(\sqrt{5}, 0)$

c) The transverse axis is horizontal and measures 6 units. *(−3, 0) and (3, 0)*

d) The conjugate axis is horizontal and has length 6 units and the focal distance is equal to 10 units.

(0, −4) and (0, 4)

9. In each of the following cases, determine the equation of the hyperbola centred at the origin.

a) $F_1(-4, 0)$ and $F_2(4, 0)$ are the foci and $A_1(-3, 0)$ and $A_2(3, 0)$ are the vertices. $\frac{x^2}{9} - \frac{y^2}{7} = 1$

b) $F_1(0, -3)$ and $F_2(0, 3)$ are the foci and $B_1(0, -2)$ and $B_2(0, 2)$ are the vertices. $\frac{x^2}{5} - \frac{y^2}{4} = -1$

c) The transverse axis is vertical and measures 8 units and the focal distance is equal to 10 units.

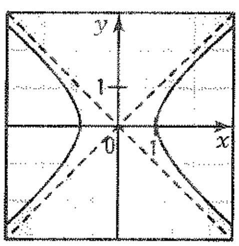
$\frac{x^2}{9} - \frac{y^2}{16} = -1$

d) (−2, 0) is a vertex of the hyperbola and one of the asymptotes has equation: $y = \frac{3}{2}x$.

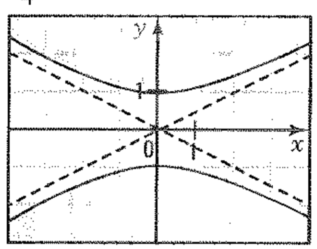
$\frac{x^2}{4} - \frac{y^2}{9} = 1$

10. Represent the following inequalities in the Cartesian plane.

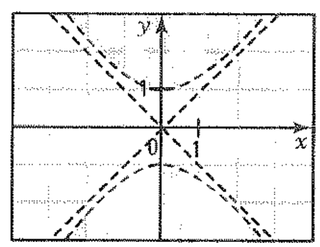
a) $x^2 - y^2 \leq 1$



b) $\frac{x^2}{4} - y^2 \leq -1$

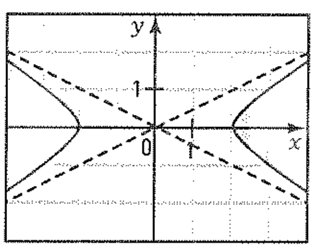


c) $x^2 - y^2 < -1$



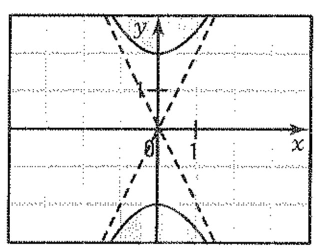
11. Describe each of the following regions using an inequality.

a)



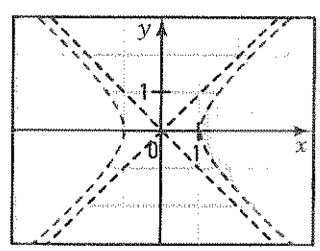
$\frac{x^2}{4} - y^2 \geq 1$

b)



$x^2 - \frac{y^2}{4} \leq -1$

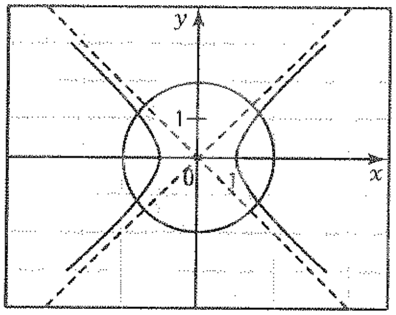
c)



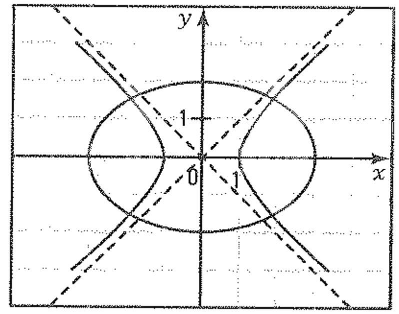
$x^2 - y^2 > 1$

12. Represent the following systems.

a) $\begin{cases} x^2 - y^2 \leqslant 1 \\ x^2 + y^2 \leqslant 4 \end{cases}$

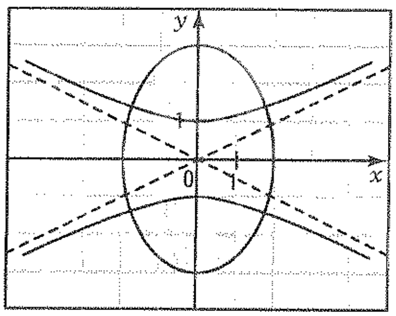


b) $\begin{cases} x^2 - y^2 \geqslant 1 \\ 4x^2 + 9y^2 - 36 \leqslant 0 \end{cases}$



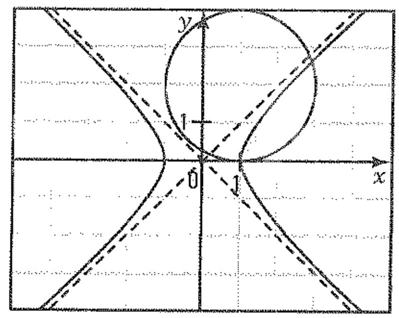
13. Describe each of the following shaded regions using a system of inequalities.

a)



$\begin{cases} \frac{x^2}{4} - y^2 \leqslant -1 \\ \frac{x^2}{4} + \frac{y^2}{9} \leqslant 1 \end{cases}$

b)



$\begin{cases} x^2 - y^2 \geqslant 1 \\ (x-1)^2 + (y-2)^2 \leqslant 4 \end{cases}$

14. Determine the coordinates of the intersection points of the hyperbola with equation: $x^2 - y^2 = 1$ and the circle with equation: $x^2 + y^2 = 7$.

$(-2, -\sqrt{3}), (-2, \sqrt{3}), (2, -\sqrt{3})$ and $(2, \sqrt{3})$

$1 + y^2 = 7 - y^2$
$2y^2 = 6$
$y^2 = 3$
$y = \pm\sqrt{3}$

15. Determine the coordinates of the intersection points of the hyperbola with equation: $x^2 - y^2 = -1$ and the line with equation: $y = 2x + 1$.

$(0, 1)$ and $\left(-\frac{4}{3}, \frac{-5}{3}\right)$

16. Determine the coordinates of the intersection points of the ellipse with equation: $x^2 + 2y^2 - 2 = 0$ and the hyperbola with equation: $x^2 - 2y^2 + 2 = 0$.

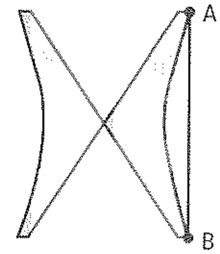
$(0,1)$ and $(0, -1)$

17. The logo of a company is represented on the right.

The curve shown is a hyperbola with equation: $16x^2 - 9y^2 = 3600$ where the measurement unit is the millimetre.

The line segment AB passes through the focus F of the hyperbola and is perpendicular to the transverse axis.

What is, rounded to the nearest hundredth of a unit, the length of line segment AB?

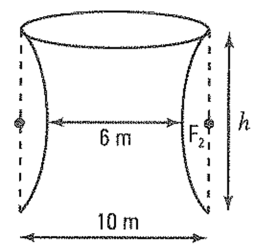


Hyperbola: $\frac{x^2}{225} - \frac{y^2}{400} = 1$; $F(25, 0)$; $A\left(25, \frac{80}{3}\right)$ ***and*** $B\left(25, \frac{-80}{3}\right)$

$m\overline{AB} = 53.33$ ***mm.***

18. The outline of a chimney shaped like a hyperbola is represented on the right. The focal distance is equal to 10 m.

Determine, to the nearest hundredth of a unit, the height h of the chimney.

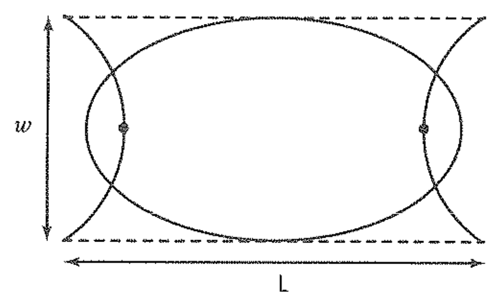


Equation of the hyperbola: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Height of the chimney: 10.67 m.

19. A mechanical part is composed of an ellipse and two arcs of hyperbola. The foci of the ellipse are the vertices of the hyperbola and the foci of the hyperbola correspond to two vertices of the ellipse.

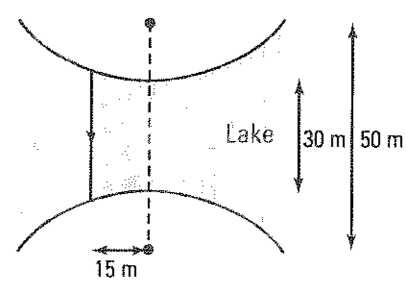
The focal distance of the ellipse is equal to 16 cm while the focal distance of the hyperbola is equal to 20 cm. The width w of the mechanical part is equal to the measure of the minor axis of the ellipse. Determine the length L of the part, to the nearest tenth of a unit.



Equation of the ellipse: $\frac{x^2}{100} + \frac{y^2}{36} = 1$; $w = 12$ ***cm; Equation of the hyperbola:*** $\frac{x^2}{64} - \frac{y^2}{36} = 1$;

length L = 22.6 cm.

20. The two shores of a lake are shaped like the two branches of a hyperbola around the area where the distance between the shores is the shortest. We observed that the shortest distance is 30 m and that the focal distance is 50 m. What distance, rounded to the nearest unit, must we travel to cross this lake in the direction parallel to the transverse axis and at a distance of 15 m from this axis?



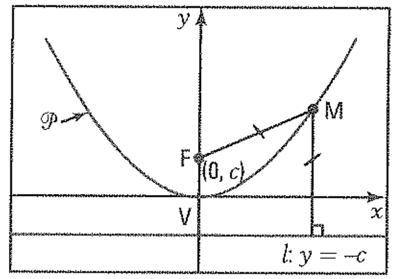
Hyperbola: $\frac{x^2}{400} - \frac{y^2}{225} = -1$.

We must travel a distance of 38 m.

7.5 Parabola

ACTIVITY 1 Parabola with vertex V(0, 0) open upwards

Consider the line l with equation: $y = -c$ and the point F(0, c), ($c > 0$). The set of points M of the plane whose distances from line l and point F are equal is a **parabola** whose **focus** is point F and whose **directrix** is line l.



a) Verify that the vertex V(0, 0) of the parabola is at the same distance from the focus F and the directrix l.

b) Justify the steps allowing us to find the equation of parabola $\mathscr{P}$.

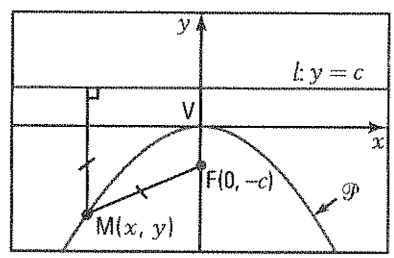
	Steps	Justifications
1.	$M(x, y) \in \mathscr{P} \Leftrightarrow d(M, F) = d(M, l)$	***Definition of parabola*** $\mathscr{P}$.
2.	$\Leftrightarrow \sqrt{x^2 + (y - c)^2} = y + c$	***Formula for the distance between 2 points and formula for the distance between a point and a line.***
3.	$\Leftrightarrow x^2 + (y - c)^2 = (y + c)^2$	***Squaring both sides.***
4.	$\Leftrightarrow x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2$	***Expansion of the squares.***
5.	$\Leftrightarrow x^2 = 4cy$	***Reduction.***

ACTIVITY 2 Parabola with vertex V(0, 0) open downwards

Show that the parabola $\mathscr{P}$ whose focus is the point F(0, $-c$), ($c > 0$) and whose directrix is the line l: $y = c$ has equation: $x^2 = -4cy$.

$M(x, y) \in \mathscr{P} \Leftrightarrow \sqrt{x^2 + (y + c)^2} = c - y \Leftrightarrow x^2 + (y + c)^2 = (c - y)^2$

$\Leftrightarrow x^2 + y^2 + 2cy + c^2 = c^2 - 2cy + y^2 \Leftrightarrow x^2 = -4cy.$

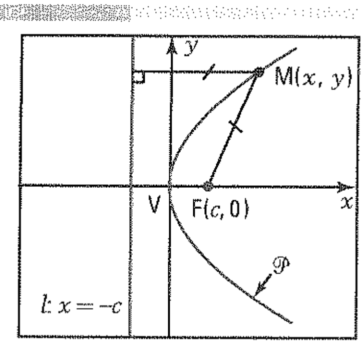


ACTIVITY 3 Parabola with vertex V(0, 0) open to the right

Show that the parabola $\mathscr{P}$ whose focus is the point F(c, 0), $c > 0$ and whose directrix is the line l: $x = -c$ has equation: $y^2 = 4cx$.

$M(x, y) \in \mathscr{P} \Leftrightarrow \sqrt{(x - c)^2 + y^2} = x + c \Leftrightarrow (x - c)^2 + y^2 = (x + c)^2$

$\Leftrightarrow x^2 - 2cx + c^2 + y^2 = x^2 + 2cx + c^2 \Leftrightarrow y^2 = 4cx.$

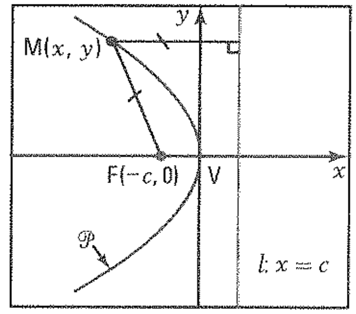


ACTIVITY 4 Parabola with vertex V(0, 0) open to the left

Show that the parabola $\mathcal{P}$ whose focus is the point $F(-c, 0)$, $(c > 0)$ and whose directrix is the line l: $x = c$ has equation: $y^2 = -4cx$.

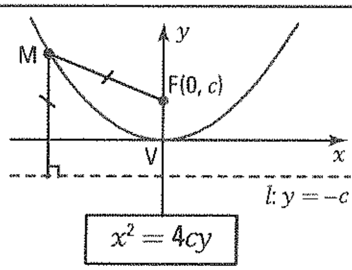
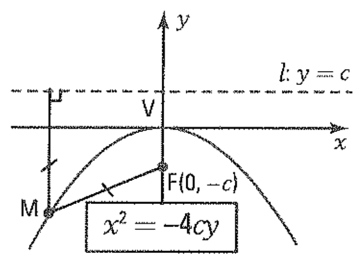
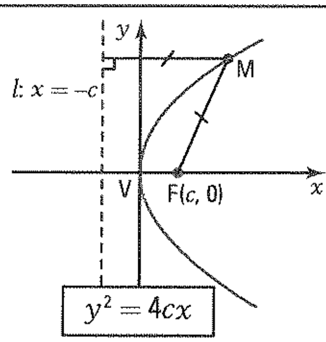
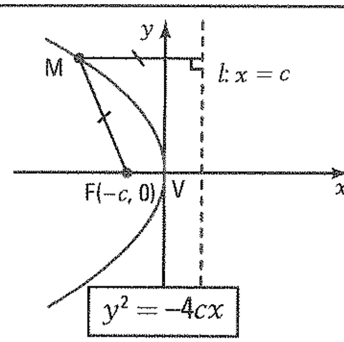
$M(x, y) \in \mathcal{P} \Leftrightarrow \sqrt{(x + c)^2 + y^2} = c - x \Leftrightarrow (x + c)^2 + y^2 = (c - x)^2$

$\Leftrightarrow x^2 + 2cx + c^2 + y^2 = c^2 - 2cx + x^2 \Leftrightarrow y^2 = -4cx.$



PARABOLA WITH VERTEX V(0, 0)

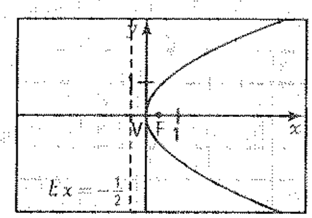
- The **parabola** with **focus** F and **directrix** l (line not passing through the focus) is the set of all points M of the plane whose distances from the focus and the directrix are equal.
- We distinguish four cases according to the concavity of the parabola with **vertex** V(0, 0).

Concave upwards	Concave downwards
 $x^2 = 4cy$	 $x^2 = -4cy$
Concave to the right	**Concave to the left**
 $y^2 = 4cx$	 $y^2 = -4cx$

- The y-axis is an **axis of symmetry** when the parabola is concave upwards or downwards. The x-axis is an **axis of symmetry** when the parabola is concave to the right or to the left.
- In all cases, we have:
  - $d(V, F) = d(V, l) = c$ $(c > 0)$
  - The axis of symmetry passes through the vertex V, the focus F and is perpendicular to the directrix l.

Ex.: The parabola with equation $y^2 = 2x$ $\left(c = \frac{1}{2}\right)$, is open to the right.

We have: $V(0, 0)$; $F\left(\frac{1}{2}, 0\right)$; $l: x = -\frac{1}{2}$;



x	0	1	2	3	4
y	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{6}$	$\pm 2\sqrt{2}$

1. For each of the following parabolas with vertex V(0, 0), determine

1. the concavity. 2. the coordinates of the focus. 3. the equation of the directrix.

a) $x^2 = 6y$
1. *Upwards*
2. *F(0; 1.5)*
3. *y = –1.5*

b) $x^2 = -4y$
1. *Downwards*
2. *F(0, –1)*
3. *y = 1*

c) $y^2 = -12x$
1. *To the left*
2. *F(–3, 0)*
3. *x = 3*

d) $y^2 = 8x$
1. *To the right*
2. *F(2, 0)*
3. *x = –2*

2. A parabola with vertex V(0, 0), open to the left, passes through point A(–5, –5). What is its equation? $y^2 = -5x$

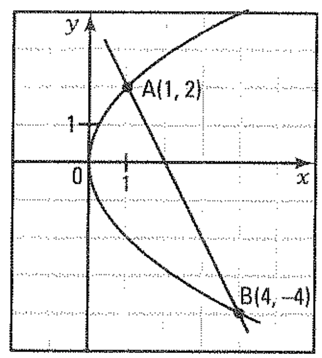
3. A parabola with vertex V(0, 0), passes through point A(4, 4). What is its equation if the parabola is

a) open upwards? $x^2 = 4y$ **b)** open to the right? $y^2 = 4x$

4. Complete the following table knowing that each parabola has vertex V(0, 0).

Equation of the parabola	Focus	Equation of the directrix	Equation of the axis of symmetry	A point on the parabola
$x^2 = -6y$	$F(-1.5, 0)$	$y = 1.5$	$x = 0$	A(6, –6)
$y^2 = 8x$	F(2, 0)	$x = -2$	$y = 0$	A(2, –4)
$x^2 = 2y$	$F\left(0, \frac{1}{2}\right)$	$y = -\frac{1}{2}$	$x = 0$	A(–4, 8)
$y^2 = -x$	$F\left(-\frac{1}{4}, 0\right)$	$x = \frac{1}{4}$	$y = 0$	A(–1, 1)

5. Consider the parabola $\mathcal{P}$ with equation: $y^2 = 4x$ and the line l with equation: $2x + y - 4 = 0$.



a) Determine algebraically the intersection points of parabola $\mathcal{P}$ and line l.

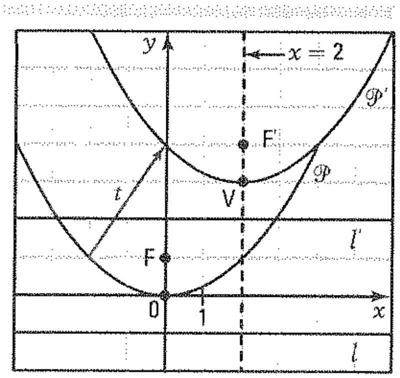
$\begin{cases} y^2 = 4x \\ y = -2x + 4 \end{cases} \Rightarrow (-2x + 4)^2 = 4x;\ 4x^2 - 20x + 16 = 0$

A(1, 2); B(4, –4)

b) Represent the parabola and line l in the Cartesian plane and verify the answers found in a).

ACTIVITY 5 Parabola with vertex V(*h*, *k*)

Consider the parabola $\mathcal{P}$ with vertex 0(0, 0), equation $x^2 = 4y$, focus F(0, 1) and directrix $l: y = -1$.



a) We apply to parabola $\mathcal{P}$, its focus F and its directrix l, the translation t defined by $t: (x, y) \rightarrow (x + 2, y + 3)$.

Draw the parabola $\mathcal{P}'$, image of parabola $\mathcal{P}$, the image F′ of the focus F and the image l' of the directrix l.

b) What is, for parabola $\mathcal{P}'$

1. its focus? *F′* 2. its directrix? *l′*

c) Determine

1. the coordinates of the vertex V of parabola $\mathcal{P}'$. $V(2, 3)$
2. the coordinates of the focus F′ of parabola $\mathcal{P}'$. $F'(2, 4)$
3. the equation of the directrix l' of parabola $\mathcal{P}'$. $y = 2$
4. the equation of the axis of symmetry of $\mathcal{P}'$. $x = 2$

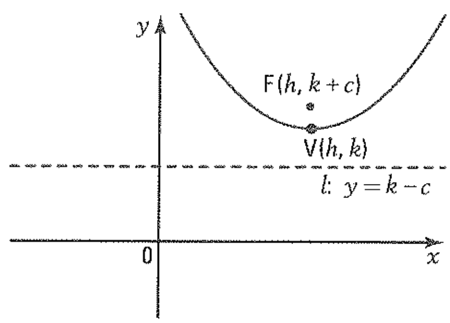
d) Deduce the equation (in the standard form) of parabola $\mathcal{P}'$ from the equation of parabola $\mathcal{P}$.

$(x + 2)^2 = 4(y + 3)$

PARABOLA WITH VERTEX V(h, k)

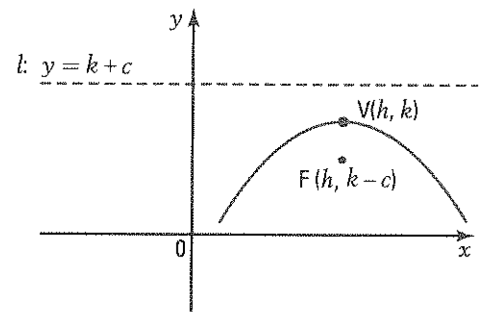
We have the following four situations:

Concave upwards



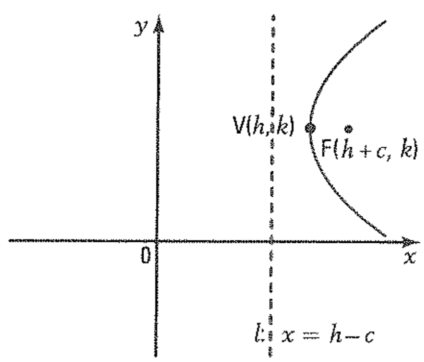
$$(x - h)^2 = 4c(y - k)$$

Concave downwards



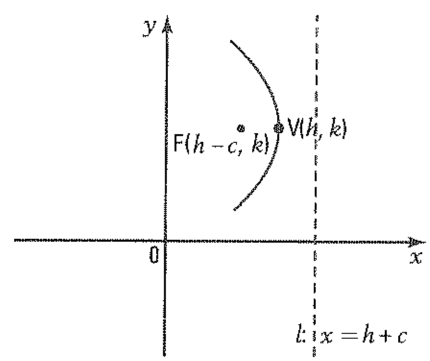
$$(x - h)^2 = -4c(y - k)$$

Concave to the right



$$(y - k)^2 = 4c(x - h)$$

Concave to the left



$$(y - k)^2 = -4c(x - h)$$

6. Complete the following table.

Equation of the parabola	Parameter c	Concavity	Coordinates of the vertex	Coordinates of the focus	Equation of the directrix
$(x+3)^2 = 4(y-1)$	*1*	*Upwards*	*V(-3, 1)*	*F(-3, 2)*	$y = 0$
$(y-2)^2 = 2(x+1)$	$\frac{1}{2}$	*To the right*	*V(-1, 2)*	$F\left(\frac{-1}{2}, 2\right)$	$x = \frac{-3}{2}$
$(x+1)^2 = -4(y+3)$	*1*	*Downwards*	*V(-1, -3)*	*F(-1, -4)*	$y = -2$
$(y+2)^2 = -2(x-1)$	$\frac{1}{2}$	*To the left*	*V(1, -2)*	$F\left(\frac{1}{2}, -2\right)$	$x = \frac{3}{2}$

7. Complete the following table.

Equation of the parabola	Parameter c	Concavity	Coordinates of the vertex	Coordinates of the focus	Equation of the directrix
$(x-1)^2 = -6(y+2)$	$\frac{3}{2}$	Downwards	V(1, –2)	$F\left(1, \frac{-7}{2}\right)$	$y = \frac{-1}{2}$
$(x+1)^2 = 4(y-3)$	1	Upwards	V(–1, 3)	*F(-1, 4)*	$y = 2$
$(y-2)^2 = -2(x+1)$	$\frac{1}{2}$	*To the left*	V(–1, 2)	$F\left(-\frac{3}{2}, 2\right)$	$x = \frac{-1}{2}$
$(y+1)^2 = 6(x-2)$	$\frac{3}{2}$	*To the right*	*V(2, -1)*	$F\left(\frac{7}{2}, -1\right)$	$x = \frac{1}{2}$

8. Determine the equation of the parabola in each of the following cases.

a) The parabola has focus F(1, 4) and directrix $l: y = 2$. $(x-1)^2 = 4(y-3)$

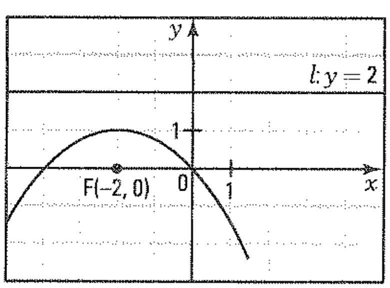
b) The parabola has vertex V(–1, 2), is open to the right and passes through point A(3, 6).
$(y-2)^2 = 4(x+1)$

c) The parabola has vertex V(1, 3), is open downwards and passes through point A(3, 1).
$(x-1)^2 = -2(y-3)$

9. In each of the following cases,

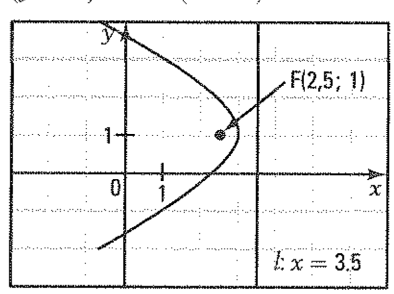
1. draw the parabola.
2. locate the focus F.
3. draw the directrix l.

a) $(x+2)^2 = -4(y-1)$



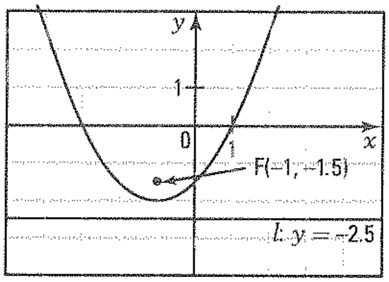
x	–5	–4	–2	0	1
y	–1.25	0	1	0	–1.25

b) $(y-1)^2 = -2(x-3)$



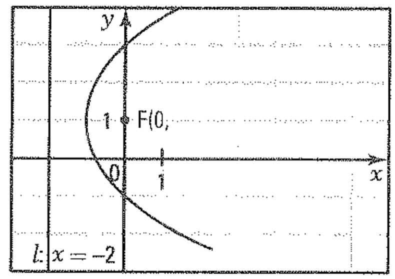
x	0	1	2.5	3	
y	–1.5 / 3.5	–1 / 3	0 / 2	1	

c) $(x+1)^2 = 2(y+2)$



x	−4	−3	−1	1	2
y	2.5	0	−2	0	2.5

d) $(y-1)^2 = 4(x+1)$



x	−1	0	1
y	1	−1 / 3	−1.8 / 3.8

ACTIVITY 6 Equation of the parabola – General form

a) Consider the parabola with equation (standard form): $(x-1)^2 = 2(y+2)$. Expand this equation in order to write it in the form $y = ax^2 + bx + c$ called general form.

$y = \frac{1}{2}x^2 - x - \frac{3}{2}$

b) A parabola has equation (general form): $y = 2x^2 - 4x + 1$.

1. Justify the steps allowing us to write the equation in the standard form.

	Steps	Justifications
	$y = 2x^2 - 4x + 1$	
1	$\Leftrightarrow y = 2(x^2 - 2x) + 1$	*We factorize the first 2 terms of the trinomial.*
2	$\Leftrightarrow y = 2(x^2 - 2x + 1) + 1 - 2$	*We complete the trinomial to obtain a perfect square.*
3	$\Leftrightarrow y = 2(x-1)^2 - 1$	*We factorize the perfect square trinomial.*
4	$\Leftrightarrow (x-1)^2 = \frac{1}{2}(y+1)$	*We isolate $(x-1)^2$ to obtain the standard form.*

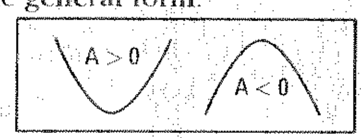
2. Determine the coordinates of the vertex V, the coordinates of the focus F and the equation of the directrix l of this parabola.

$V(1, -1)$; $F\left(1, -\frac{7}{8}\right)$; $y = \frac{-9}{8}$

GENERAL EQUATION OF THE PARABOLA

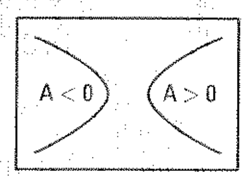
- By expanding the equation $(x-h)^2 = 4c(y-k)$ of a parabola open upwards or the equation $(x-h)^2 = -4c(y-k)$ of a parabola open downwards, we obtain the general form:

$$y = Ax^2 + Bx + C \quad (A \neq 0).$$



- By expanding the equation $(y-k)^2 = 4c(x-h)$ of a parabola open to the right or the equation $(y-k)^2 = -4c(x-h)$ of a parabola open to the left, we obtain the general form:

$$x = Ay^2 + By + C \quad (A \neq 0).$$



10. Write the equations of the following parabolas in the general form.

a) $(x-1)^2 = 2(y+1)$ ***$y = \frac{x^2}{2} - x - \frac{1}{2}$*** b) $(x+2)^2 = -4(y-1)$ ***$y = -\frac{x^2}{4} - x$***

c) $(y+3)^2 = \frac{3}{2}(x-4)$ ***$x = \frac{2}{3}y^2 + 4y + 10$*** d) $(y-2)^2 = \frac{-1}{2}(x+2)$ ***$x = -2y^2 + 8y - 10$***

11. Write the equations of the following parabolas in the standard form.

a) $y = x^2 + 2x + 3$ ***$(x+1)^2 = (y-2)$*** b) $x = y^2 - 6y + 10$ ***$(y-3)^2 = (x-1)$***

c) $y = -x^2 + 4x + 6$ ***$(x-2)^2 = -(y-10)$*** d) $x = -y^2 - 2y + 1$ ***$(y+1)^2 = -(x-2)$***

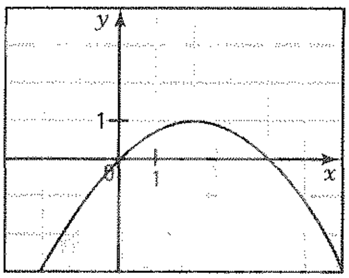
12. Determine the coordinates of the focus F and the equation of the directrix l of the following parabolas.

a) $y = x^2 - 2x - 1$; ***$(x-1)^2 = y + 2$; $F\left(1, -\frac{7}{4}\right)$; l: $y = -\frac{9}{4}$***

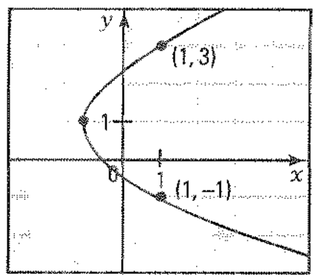
b) $x = \frac{1}{4}y^2 - y - 1$; ***$(y-2)^2 = 4(x+2)$; $F(-1, 2)$; l: $x = -3$***

13. Represent the solution set of the following inequalities in the Cartesian plane.

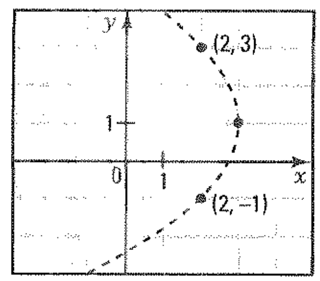
a) $(x-2)^2 \leqslant -4(y-1)$



b) $(y-1)^2 \geqslant 2(x+1)$

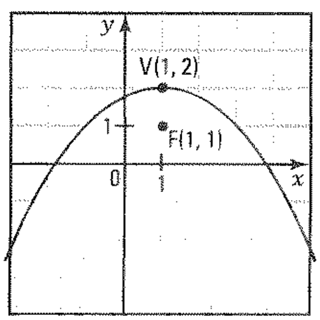


c) $(y-1)^2 < -4(x-3)$



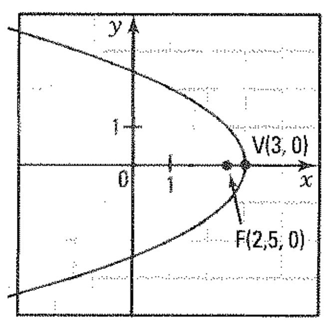
14. For each of the following regions, determine the inequality that defines it.

a)



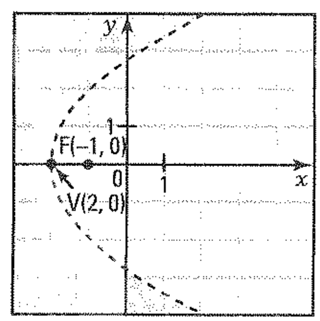
$(x-1)^2 \leqslant -4(y-2)$

b)



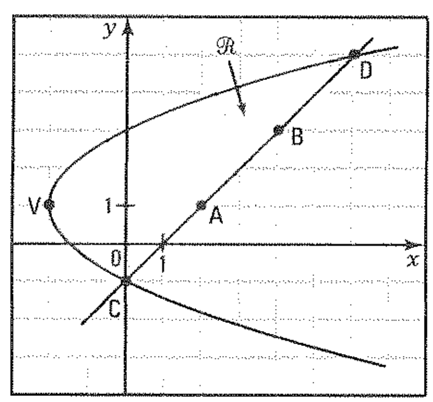
$y^2 \leqslant -2(x-3)$

c)



$y^2 > 4(x+2)$

15. Consider the parabola $\mathscr{P}$ with vertex V(−2, 1) and focus $F\left(\frac{-3}{2}, 1\right)$ and the line l passing through points A(2, 1) and B(4, 3).



a) Find the intersection points C and D of parabola $\mathscr{P}$ and line l. ***C(0, −1), D(6, 5)***

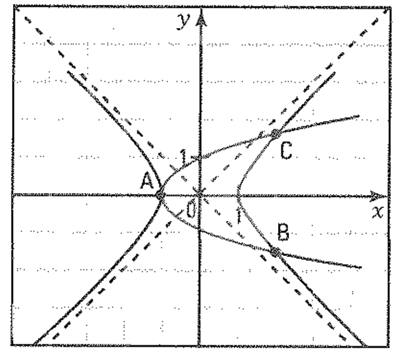
$\mathscr{P}$: $(y-1)^2 = 2(x+2)$; l: $y = x - 1$

b) Draw parabola $\mathscr{P}$ and line l and verify the results found in a).

c) Consider the closed region $\mathscr{R}$ whose boundary is parabola $\mathscr{P}$ and line l.

Describe region $\mathscr{R}$ using a system of inequalities. ***$\begin{cases}(y-1)^2 \leqslant 2(x+2) \\ y \geqslant x - 1\end{cases}$***

16. Consider the hyperbola $\mathscr{H}$ with equation: $x^2 - y^2 = 1$ and the parabola $\mathscr{P}$ with equation: $x + 1 = y^2$.



a) Find the intersection points of the hyperbola and the parabola.

$x^2 - (x + 1) = 1 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x = -1$ or $x = 2$

There are 3 intersection points.

$A(-1, 0)$, $B(2, -\sqrt{3})$, $C(2, \sqrt{3})$

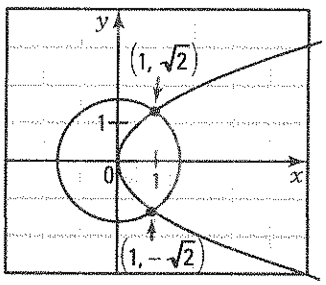
b) Represent, in the Cartesian plane, the region defined by the system $\begin{cases} x^2 - y^2 \leqslant 1 \\ x + 1 \geqslant y^2 \end{cases}$.

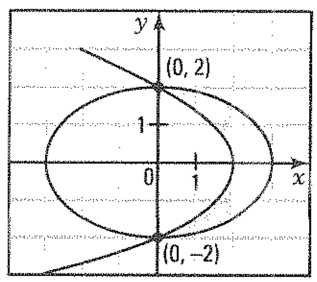
17. In each of the following cases, represent the region defined by the system.

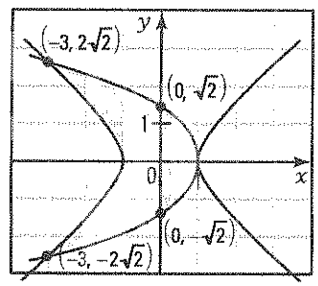
a) $\begin{cases} x^2 + y^2 \leqslant 3 \\ y^2 \leqslant 2x \end{cases}$

b) $\begin{cases} y^2 \geqslant -2(x-2) \\ 4x^2 + 9y^2 - 36 \leqslant 0 \end{cases}$

c) $\begin{cases} x - 1 \leqslant \frac{-1}{2} y^2 \\ x^2 - y^2 \leqslant 1 \end{cases}$

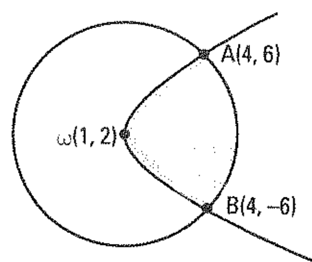






18. Consider the circle $\mathscr{C}$ and the parabola $\mathscr{P}$ on the right. The vertex of the parabola is the centre $\omega(1, 2)$ of the circle. The points A(4, 6) and B(4, –6) are the intersection points of the circle and the parabola.

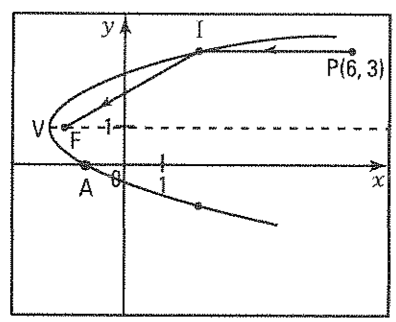
Describe, using a system, the shaded region.



$$\begin{cases} (y - 1)^2 + (y - 2)^2 \leqslant 25 \\ (y - 2)^2 \leqslant \frac{16}{3}(x - 1) \end{cases}$$

19. The parabola on the right, open to the right with vertex V(–2, 1) crosses the x-axis at point A(–1, 0).

Calculate the distance traveled by a light ray going from point P(6, 3) in a direction parallel to the x-axis, hitting the parabola at point I and reflected at the focus F.



Parabola: $(y - 1)^2 = x + 2$; $c = \frac{1}{4}$; $F\left(-\frac{7}{4}, 1\right)$.

$I(2, 3)$: $m\overline{PI} = 4$ u; $m\overline{IF} = \frac{17}{4}$ u ; distance traveled = 8.25 u.

7.6 Problems on conics

1. Find the equation in the standard form of each of the following conics.

a) Ellipse with foci $F_1(-8, 0)$ and $F_2(8, 0)$ whose major axis measures 20 units. $\frac{x^2}{100} + \frac{y^2}{36} = 1$

b) Hyperbola with foci $F_1(-10, 0)$ and $F_2(10, 0)$ whose transverse axis has a length of 12 units.

$\frac{x^2}{36} - \frac{y^2}{64} = 1$

c) Circle centred at $O(0, 0)$ passing through $A(-2, 3)$. $x^2 + y^2 = 13$

d) Parabola with vertex $V(0, 0)$ and focus $F(0, -3)$. $x^2 = -12y$

2. For each of the conics defined by the following equations, describe the conic by giving:
- for a circle, the centre and the radius,
- for an ellipse, the coordinates of its foci and its vertices,
- for a hyperbola, the coordinates of its foci, its vertices and the equations of the asymptotes,
- for a parabola, the coordinates of its vertex, its focus and the equation of the directrix.

a) $x^2 + y^2 - 5 = 0$. *Circle of radius $\sqrt{5}$ with centre $O(0,0)$*

b) $x^2 + 4y^2 - 16 = 0$.

$\frac{x^2}{16} + \frac{y^2}{4} = 1$; *Ellipse centred at the origin; vertices: $(-4, 0)$, $(4, 0)$, $(0, -2)$, $(0, 2)$;*

foci: $(-\sqrt{12}, 0)$ and $(\sqrt{12}, 0)$

c) $4x^2 - 9y^2 - 36 = 0$.

$\frac{x^2}{9} - \frac{y^2}{4} = 1$; *Hyperbola centred at the origin; vertices: $(-3, 0)$, $(3, 0)$;*

foci: $(-\sqrt{13}, 0)$ and $(\sqrt{13}, 0)$ and asymptotes: $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$.

d) $y^2 + 2x - 2y + 7 = 0$.

$(y - 1)^2 = -2(x + 3)$; Parabola with vertex $(-3, 1)$ open to the left;

focus: $\left(-\frac{7}{2}, 1\right)$ and directrix: $x = -\frac{5}{2}$.

3. In each of the following cases, name the geometric locus and give its equation.

a) Set of points $M(x, y)$ located at a distance of 3 units from the origin. *Circle: $x^2 + y^2 = 9$.*

b) Set of points $M(x, y)$ whose distances from the point $(-1, 2)$ and the line with equation $x = 3$ are equal. *Parabola: $(y - 2)^2 = -8(x - 1)$.*

c) Set of points $M(x, y)$ such that the absolute value of the difference of the distances from point M to points $(-5, 0)$ and $(5, 0)$ is equal to 8 units.

Hyperbola: $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

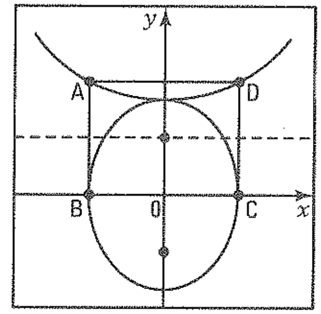
d) Set of points $M(x, y)$ such that the sum of the distances from point M to points $(0, -4)$ and $(0, 4)$ is equal to 10 units.

Ellipse: $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

4. The ellipse centred at the origin on the right has a major axis measuring 10 units and a minor axis measuring 8 units.

The vertex of the parabola on the right coincides with one of the vertices of the ellipse and its directrix passes through one of the foci of the ellipse.

Calculate the area of rectangle ABCD if line segment BC corresponds to the horizontal minor axis of the ellipse.

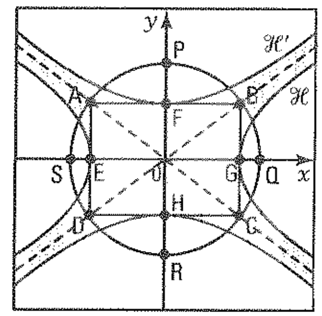


Ellipse: $\frac{x^2}{16} + \frac{y^2}{25} = 1$*; Parabola:* $x^2 = 8(y - 5)$

B(−4, 0); C(4, 0); A(−4, 7); D(4, 7).

Area of rectangle ABCD = 56 u^2.

5. Consider the circle centred at 0 passing through point A(−4, 3) and the rectangle ABCD whose sides are parallel to the axes. Consider the hyperbola $\mathcal{H}$ whose transverse axis is on the x-axis, whose vertices are points E and G and whose foci are points S and Q and the hyperbola $\mathcal{H}'$ whose transverse axis is on the y-axis, whose vertices are points F and H and whose foci are points P and R.



a) Determine the system of inequalities representing the shaded region.

$\mathcal{H}: \frac{x^2}{16} - \frac{y^2}{9} = 1$; $\mathcal{H}': \frac{x^2}{16} - \frac{y^2}{9} = -1$; $\begin{cases} \frac{x^2}{16} - \frac{y^2}{9} \leqslant 1 \\ \frac{x^2}{16} - \frac{y^2}{9} \geqslant -1 \end{cases}$

b) What can be said about the supports of the diagonals of rectangle ABCD. Justify your answer.

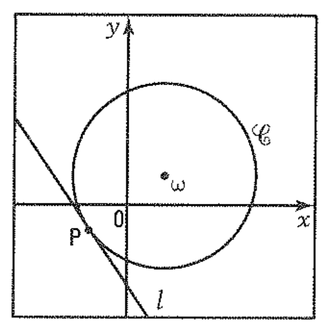
AC: $y = -\frac{3}{4}x$ *(shared asymptote for hyperbolas* $\mathcal{H}$ *and* $\mathcal{H}'$*)*

BD: $y = \frac{3}{4}x$ *(shared asymptote for hyperbolas* $\mathcal{H}$ *and* $\mathcal{H}'$*)*

6. A circle $\mathscr{C}$ centred at $\omega(-1, 2)$ passes through point A(2, 6). Find the equation of the line l tangent to the circle at point P(3, −1).

$\mathscr{C}: (x + 1)^2 + (y - 2)^2 = 25$; $l: y = \frac{4}{3}x - 5$

7. Circle $\mathscr{C}$ centred at $\omega(4, 3)$ on the right is tangent to line l with equation: $4x + 3y + 25 = 0$.



a) Find the equation of circle $\mathscr{C}$.

$r = d(\omega, l) = \frac{|4 \times 4 + 3 \times 3 + 25|}{\sqrt{4^2 + 3^2}} = \frac{50}{5} = 10$

$\mathscr{C}: (x - 4)^2 + (y - 3)^2 = 100$

b) Find the coordinates of the point of tangency P.

P(−4, −3)

8. Consider triangle ABC having vertices A(–5, 10), B(–7, –4) and C(9, 8).

Find the equation of the circumscribed circle of triangle ABC.

Recall that: The circumscribed circle's centre is the intersection point of the perpendicular bisectors of the sides of the triangle.

l_1 (perpendicular bisector of $\overline{AB}$): $y = -\frac{1}{7}x + \frac{15}{7}$; l_2 (perpendicular bisector of $\overline{AC}$): $y = 7x - 5$;

ω (centre of circumscribed circle) = (1, 2); radius of circumscribed circle = $d(\omega, A) = 10$.

Circumscribed circle: $(x - 1)^2 + (y - 2)^2 = 100$.

9. What is the intersection point of the directrices of parabolas $\mathcal{P}_1: (y - 1)^2 = -8(x + 1)$ and $\mathcal{P}_2: (x + 3)^2 = 4(y + 2)$?

$l_1: x = 1$; $l_2: y = -3$ thus the intersection point is P(1, –3).

10. The difference between the radii of two concentric circles is 3 cm.

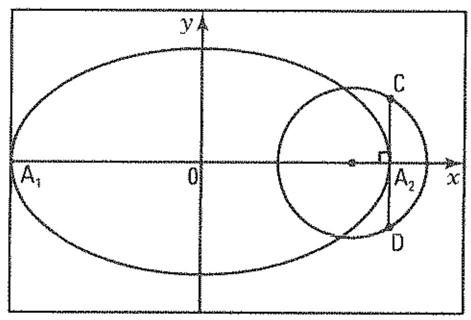
If the equation of the larger circle is: $x^2 + y^2 + 6x - 4y - 36 = 0$, determine the equation of the smaller circle.

Large circle: $(x + 3)^2 + (y - 2)^2 = 49$; small circle: $(x + 3)^2 + (y - 2)^2 = 16$

11. The major and the minor axes of the ellipse centred at the origin shown on the right measure 10 and 6 units respectively.

A circle of radius 2 units centred at one of the foci of the ellipse was drawn.

Determine the length of the chord CD knowing that $\overline{CD} \perp \overline{A_1A_2}$.

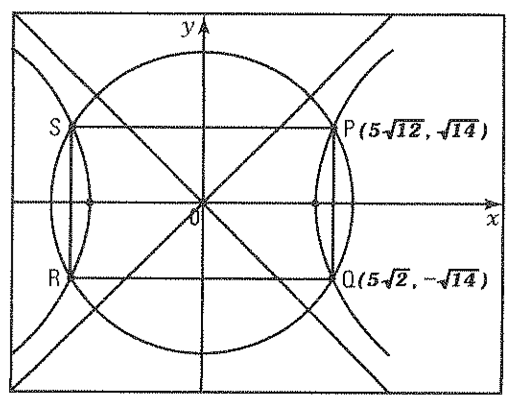


Ellipse: $\frac{x^2}{25} + \frac{y^2}{9} = 1$; Focus: F(4, 0); Circle: $(x - 4)^2 + y^2 = 4$

$C(5, \sqrt{3})$; $D(5, -\sqrt{3})$; $m\overline{CD} = 2\sqrt{3}$

12. The asymptotes of the hyperbola centred at the origin shown on the right are the lines with equations: $y = x$ and $y = -x$. Its transverse axis measures 12 units. The circle on the right, of radius 8 units and centred at the origin, intersects the hyperbola at four points P, Q, R and S.

Calculate the area of rectangle PQRS.



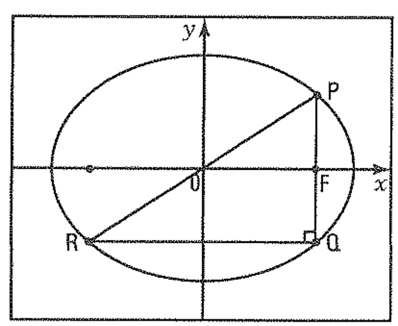
Hyperbola: $x^2 - y^2 = 36$. Circle: $x^2 + y^2 = 64$

$P(5\sqrt{2}, \sqrt{14})$; $Q(5\sqrt{2}, -\sqrt{14})$; $R(-5\sqrt{2}, -\sqrt{14})$;

$S(-5\sqrt{2}, \sqrt{14})$; $m\overline{PQ} = 2\sqrt{14}$; $m\overline{RQ} = 10\sqrt{2}$;

Area PQRS = $40\sqrt{7}$ u²

13. The major and the minor axes of the ellipse on the right measure 10 units and 6 units respectively. Calculate the area of the right triangle PQR inscribed inside the ellipse knowing that the side PQ passes through one of the foci of the ellipse.

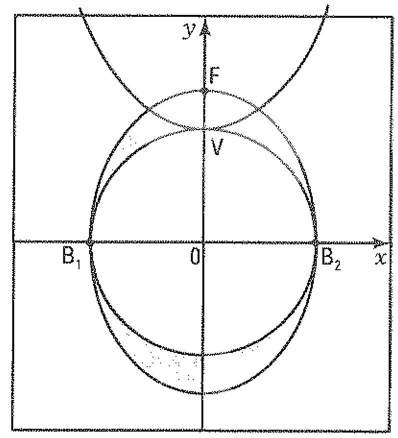


Ellipse: $\frac{x^2}{25} + \frac{y^2}{9} = 1$; F(4, 0); $P\left(4, \frac{9}{5}\right)$

$Q\left(4, \frac{-9}{5}\right)$; $R\left(-4, \frac{-9}{5}\right)$; $m\overline{PQ} = \frac{18}{5}$; $m\overline{RQ} = 8$; Area $\triangle PQR = \frac{36}{5} = 7.2$ u²

14. On the figure on the right, the circle centred at the origin passes through the vertex V of the parabola and the ellipse passes through its focus F. The minor axis B_1B_2 of the ellipse is a diameter for the circle.

If the equation of the parabola is $x^2 = 4(y - 3)$, describe the shaded region using a system of inequalities.

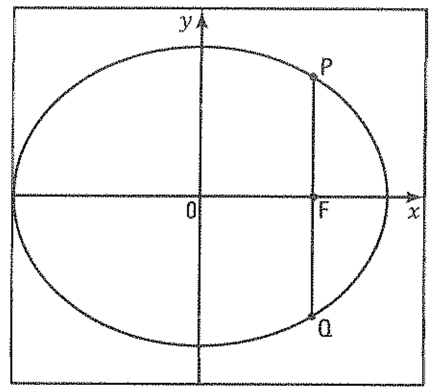


V(0, 3); F(0, 4); Circle: $x^2 + y^2 = 9$

Ellipse: $\frac{x^2}{9} + \frac{y^2}{16} = 1$. $\begin{cases} x^2 + y^2 \geq 9 \\ \frac{x^2}{9} + \frac{y^2}{16} \leq 1. \end{cases}$

15. In an elliptical shaped pool, two points P and Q on the side of this pool are joined by a cable. This cable passes through a focus of the ellipse and is perpendicular to its major axis.

Knowing that the major axis and the minor axis measure 200 m and 160 m respectively, calculate the length of the cable when it is stretched.

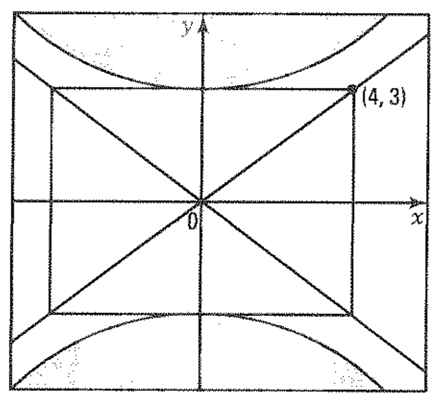


Ellipse: $\frac{x^2}{100^2} + \frac{y^2}{80^2} = 1$; F(60, 0); P(60, 64); Q(60, −64);

$m\overline{PQ}$ = 128 m.

16. The interior region of a hyperbola centred at the origin has been shaded.

Knowing that one of the asymptotes of the hyperbola passes through point (4, 3) and that the distance between the vertices of the hyperbola is equal to 6 units, describe the shaded region using an inequality.



$\frac{x^2}{16} - \frac{y^2}{9} \leq -1$

17. Consider an ellipse centred at the origin with equation: $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Determine the equation of each of the parabolas whose vertex and focus are respectively the foci of the ellipse.

Parabola: vertex $F_1(-3, 0)$ and focus $F_2(3, 0)$; $(x + 3) = 24y^2$

Parabola: vertex $F_2(3, 0)$ and focus $F_1(-3, 0)$; $(x - 3) = -24y^2$

Evaluation 7

1. In each of the following cases, name the geometric locus and give its equation.

a) The set of points $M(x, y)$ located at a distance of 3 units from the origin.
Circle: $x^2 + y^2 = 9$.

b) The set of points $M(x, y)$ whose distances from the point $(-2, 1)$ and the line with equation $x = 2$ are equal. ***Parabola: $(y - 1)^2 = -8x$***

c) The set of points $M(x, y)$ such that the sum of the distances from point $M(x, y)$ to the points $(0, -3)$ and $(0, 3)$ is equal to 10.
Ellipse: $\frac{x^2}{16} + \frac{y^2}{25} = 1$

d) The set of points $M(x, y)$ such that the absolute value of the difference of the distances to the points $(0, -10)$ and $(0, 10)$ is equal to 12.
Hyperbola: $\frac{x^2}{64} - \frac{y^2}{36} = -1$

e) The set of points $M(x, y)$ whose distances from the point $(3, -4)$ and the line $y = 2$ are equal. ***Parabola: $(x - 3)^2 = -12(y + 4)$.***

2. For each of the conics defined by the following equations, describe the conic by giving
- for a circle, the centre and the radius,
- for an ellipse, the coordinates of its foci and its vertices;
- for a hyperbola, the coordinates of its foci, its vertices and the equations of the asymptotes,
- for a parabola, the coordinates of its vertex, its focus and the equation of its directrix.

a) $(y - 3)^2 = 8(x - 1)$. ***Parabola with vertex (1, 3) open to the right, focus F(3, 3), directrix $x = -2$.***

b) $\frac{x^2}{3} + \frac{y^2}{3} = 1$ ***Circle of radius $\sqrt{3}$ centred at O(0, 0).***

c) $x^2 - 4y^2 = 4$. ***Hyperbola with vertices (−2, 0) and (2, 0), foci $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$, asymptotes: $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$.***

d) $9x^2 - 25y^2 - 225 = 0$. ***Ellipse with vertices (−5, 0), (5, 0), (0, −3) and (0, 3) and foci (−4, 0) and (4, 0).***

3. Give the equation of each of the following conics.

a) Ellipse with foci $F_1(0, -12)$ and $F_2(0, 12)$ and minor axis equal to 10 units. ***$\frac{x^2}{25} + \frac{y^2}{169} = 1$***

b) Hyperbola with foci $F_1(0, -13)$ and $F_2(0, 13)$ and transverse axis equal to 10 units.
$\frac{x^2}{144} - \frac{y^2}{25} = -1$

c) Circle centred at the origin and passing through point $A(-2, 4)$. ***$x^2 + y^2 = 20$***

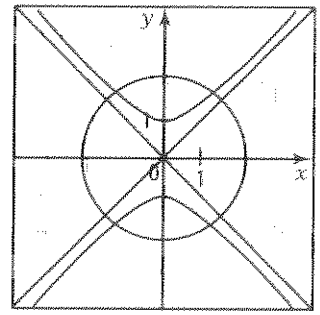
d) Parabola with vertex $V(-3, 2)$ whose directrix is the line with equation $x = -6$.
$(y - 2)^2 = 12(x + 3)$

4. A circle $\mathscr{C}$ centred at the origin is tangent to the line l: $3x - 2y - 13 = 0$.

a) Find the equation of the circle. $d(O, l) = \frac{|-13|}{\sqrt{13}} = \sqrt{13}$. $\mathscr{C}: x^2 + y^2 = 13$.

b) Find the coordinates of the point of tangency A. $A(3, -2)$

5. Consider the hyperbola $\mathscr{H}$ with equation: $x^2 - y^2 = -1$ and the circle $\mathscr{C}$ with equation: $x^2 + y^2 = 5$.



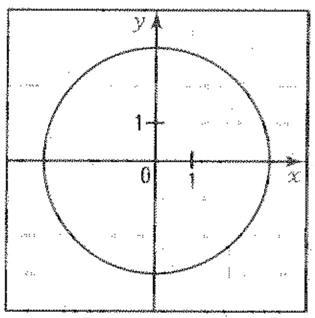
a) Find the intersection points of hyperbola $\mathscr{H}$ and circle $\mathscr{C}$.
$(-\sqrt{2}, -\sqrt{3}), (-\sqrt{2}, \sqrt{3}), (\sqrt{2}, \sqrt{3}), (\sqrt{2}, -\sqrt{3})$

b) Represent hyperbola $\mathscr{H}$ and circle $\mathscr{C}$ in the Cartesian plane.

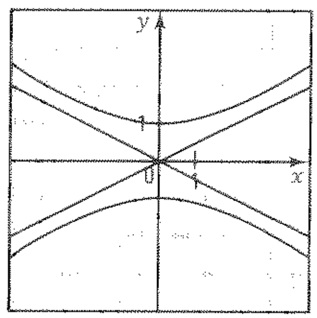
c) Colour, in the Cartesian plane, the region defined by the system: $\begin{cases} x^2 - y^2 \geq -1 \\ x^2 + y^2 \leq 5 \end{cases}$.

6. Represent the following regions in the Cartesian plane.

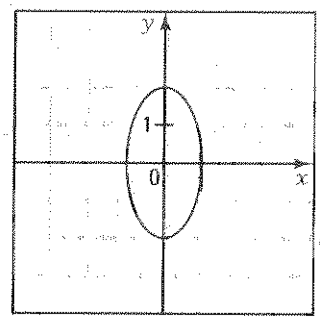
a) $x^2 + y^2 - 9 \leq 0$



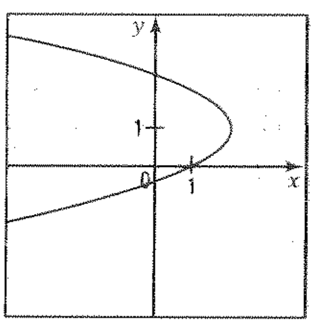
b) $x^2 - 4y^2 + 4 \geq 0$



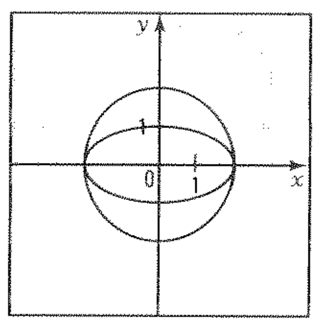
c) $4x^2 + y^2 - 4 \geq 0$



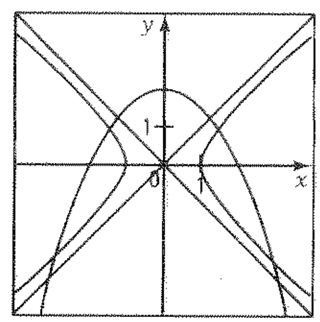
d) $(y - 1)^2 \leq -(x - 2)$



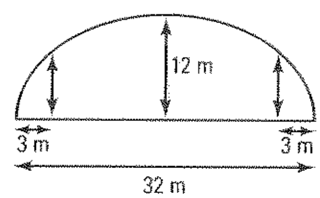
e) $\begin{cases} x^2 + 4y^2 - 4 \geq 0 \\ x^2 + y^2 - 4 \leq 0 \end{cases}$



f) $\begin{cases} y - 2 \leq -\frac{1}{2}x^2 \\ x^2 - y^2 \geq 1 \end{cases}$



7. The roof of a tunnel is shaped like a half-ellipse. If traffic is not allowed at a distance of 3 m from the edges of the tunnel, determine, to the nearest unit, the maximum height allowed for a vehicle entering this tunnel.



***Ellipse*:** $\frac{x^2}{256} + \frac{y^2}{144} = 1$; **7 m**

SYMBOLS

$\mathbb{N}$	set of natural numbers
$\mathbb{N}^*$	set of nonnegative natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{Z}_-$	set of nonpositive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Q}'$	set of irrational numbers
$\mathbb{R}$	set of real numbers
$\in$	belongs to
$\subset$	is a subset of
$\notin$	does not belong to
$\not\subset$	is not a subset of
$=$	is equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$<$	is less than
$>$	is greater than
$\leqslant$	is less than or equal to
$\geqslant$	is greater than or equal to
$\forall x$	for all x
$\Rightarrow$	logically implies
$\Leftrightarrow$	is logically equivalent
$[a, b]$	closed interval
$[a, b[$	left-closed and right-open interval
$]a, b]$	left-open and right-closed interval
$]a, b[$	open interval
$]-\infty, a]$	left-unbounded and right-closed interval
$]-\infty, a[$	left-unbounded and right-open interval
$[a, +\infty[$	left-closed and right-unbounded interval
$]a, +\infty[$	left-open and right-unbounded interval
$\varnothing$	empty set
$\sqrt[n]{a}$	n^{th} root of real number a
Δ	discriminant
S	Set of solutions
AB	line AB
$\overline{AB}$	line segment AB
$m\overline{AB}$	measure of line segment AB
$\cong$	is congruent to
$\angle$A0B	angle A0B
m $\angle$A0B	measure of angle A0B
$\perp$	is perpendicular to

//	is parallel to
$\triangle ABC$	triangle ABC
$M(x, y)$	coordonates of point M
$\sim$	is similar to
R^{-1}	inverse of relation R
$f(x)$	image of x by function f
dom f	domain of a function f
ran f	range of function a f
max f	maximum of a function f
min f	minimum of a function f
$g \circ f$	composition of f by g
$\|a\|$	absolute value of a
$[a]$	greatest integer of a
$\log_c$	logarithm in base c
log	logarithm in base 10
In	logarithm in base e
sin A	sine of angle A
cos A	cosine of angle A
tan A	tangent of angle A
sec A	secant of angle A
csc A	cosecant of angle A
cot A	cotangent of angle A
$^\circ$	degree
gr	gradian
rd	radian
$P(t)$	trigonometric point
p	period
F	frequency
arcsin	arcsine
arccos	arccosine
arctan	arctangent
Sin	principal sine
Cos	principal cosine
Tan	principal tangent
$\overrightarrow{AB}$	vector $\overrightarrow{AB}$
$\\|\overrightarrow{AB}\\|$	norm of vector $\overrightarrow{AB}$
$\vec{0}$	zero vector
$\theta_{\overrightarrow{AB}}$	orientation of vector $\overrightarrow{AB}$
$\vec{u} \cdot \vec{v}$	scalar product of two vectors $\vec{u}$ and $\vec{v}$
$\vec{u}_{\vec{v}}$	orthogonal projection of vector $\vec{u}$ onto vector $\vec{v}$

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