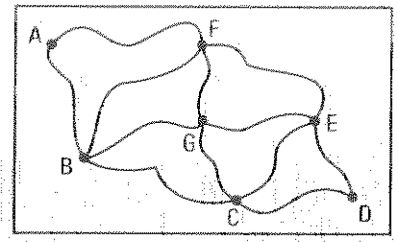
Chapter 2

Graphs

CHALLENGE 2

1. Villages A, B, C, D, E, F and G in a region are connected by roads. The map of this region is represented on the right.



a) Is it possible, when we start from a village, to follow each road of this region once and only once and then return to the village where we started? Justify your answer.

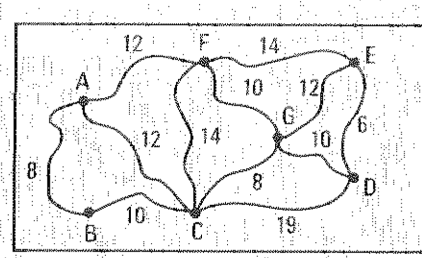
Yes, because each vertex of the graph has even degree.

b) Give one possible route when we start from village A.

A F E D C E G C B G F B A

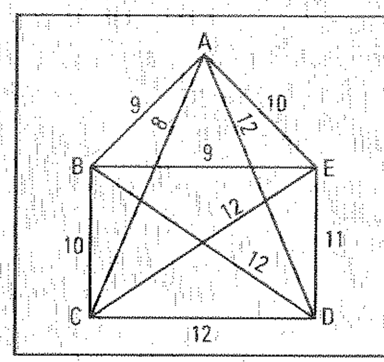
2. Each number on the map on the right represents the distance (in km) between two villages. What is the shortest route from village A to village D?

ACGD (30 km length)



3. A real estate promoter has built 5 cottages A, B, C, D and E in a mountainous region. The numbers on the map on the right indicate the cost, in thousands of dollars, of building a road connecting two cottages. What are the roads that the promoter should build if he wants to minimize the costs while ensuring that no cottage will remain isolated?

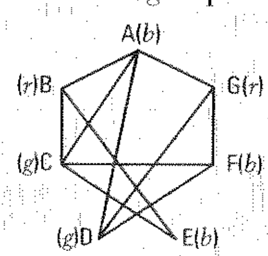
Roads DE, EB, BA and AC. Total cost $37 000.



4. Six students are working in a lab. Their teacher wants to form groups to work on a project. The teacher believes that some students cannot be in the same group. The table on the right indicates, for each student, the incompatibilities, in other words the students with whom he cannot work on the project. Indicate the minimal number of groups that the teacher can form and the members in each group.

3 groups.

{A, F, E} {B, G} and {C, D}

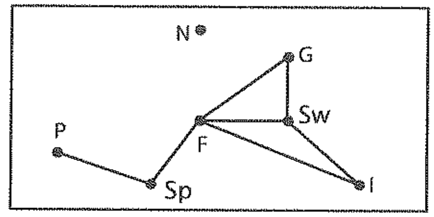


Student	Incompatibilities
A	B, C, D and G
B	A, C and E
C	A, B, E and F
D	A, F and G
E	B, C
F	C, D and G
G	A, D and F

2.1 Graphs

ACTIVITY 1 Representation of a situation using a graph

Consider a group of seven countries: Germany (G), Spain (Sp), France (F), Italy (I), Norway (N), Portugal (P) and Switzerland (Sw). These seven countries are represented on the right by points called vertices. To indicate that two countries share a common border, a line called edge is drawn. The figure on the right allowing us to visualize this situation is called a graph.



a) The number of vertices in a graph is called its order. What is the order of the graph above? *7*

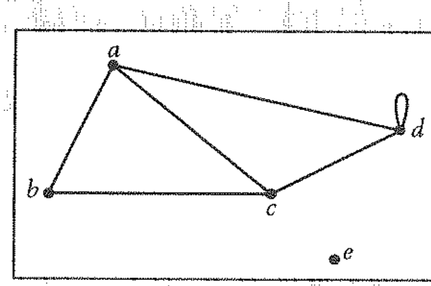
b) Two vertices are adjacent when they are connected by an edge.
1. Are vertices F and P adjacent? *No*
2. Are vertices F and I adjacent? *Yes*
3. Can we find a vertex adjacent to vertex N? *No*

c) The number of different edges reaching a vertex is called the degree of this vertex. Determine the degree of each vertex:
G: *2* Sp: *2* F: *4* I: *2* N: *0* P: *1* Sw: *3*

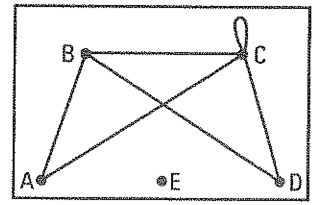
d)
1. Which country has the greatest number of common borders? *France*
2. Which country is isolated, having no common border? *Norway*

GRAPH

- A graph allows us to illustrate the elements of a set using points called vertices and relationships between these elements using lines called edges.
- An edge between a vertex a and a vertex b is written $\{a,b\}$ or $\{b,a\}$ or simply ab or ba since the order in which the vertices are written is not important.
  An edge connecting a vertex to itself is called a loop of the graph.
- Two edges are called adjacent when they are connected by an edge.
- The number of vertices in a graph is called the order of the graph.
- The degree of a vertex is equal to the number of times it is reached by an edge.
  The degree of vertex A is written: $d(A)$.
- The degree of a vertex with a loop is at least equal to 2.
  Ex.: Consider the set of vertices $S = \{a, b, c, d, e\}$
  and the graph $G = \{ab, bc, ca, ad, cd, dd\}$.
  - Vertices a and b are adjacent while vertices b and d are not adjacent.
  - The order of the graph is equal to 5.
  - The degree of vertex a is equal to 3, the degree of vertex d is equal to 4 (the loop at d counts as 2 edges), the degree of vertex e is equal to 0 because vertex e is isolated since there are no vertices in the graph adjacent to it.



1. Consider the graph on the right.



a) Name the edges. *AB, BC, CD, AC, BD and CC.*

b) Is the intersection point of edges AC and BD a vertex? *No*

c) Find two non adjacent vertices.

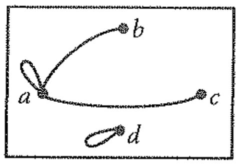
A and D for example.

d) What is the order of the graph? *5*

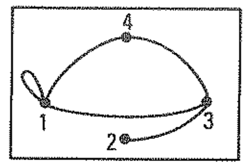
e) Find the degree of each vertex. *A: 2; B: 3; C: 5; D: 2; E: 0.*

2. Represent the graph described by

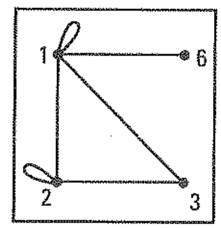
a) $S = \{a, b, c, d\}$
$G = \{aa, ab, ac, dd\}$



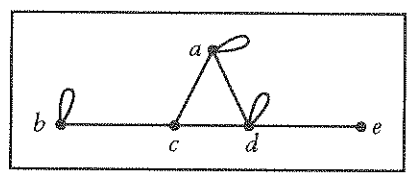
b) $S = \{1, 2, 3, 4\}$
$G = \{\{1, 1\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$



3. Consider the set $S = \{1, 2, 3, 6\}$. Draw the graph G of this set knowing that a number x in S is related to a number y in S when $xy \leqslant 6$.



4. Consider the graph on the right.



a) Name all the edges.

aa, ac, ad, bc, bb, de, dd, cd

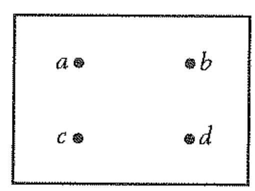
b) Find the degree of each vertex.

a: *4* b: *3* c: *3* d: *5* e: *1*

c) Verify that the sum of the degrees of all the vertices is equal to twice the total number of edges in the graph.

The sum of the degrees is equal to 16; the total number of vertices is equal to 8.

5. Consider the set of vertices on the right.



a) What is the sum of the degrees of all the vertices if we draw

1. a first edge? *2*
2. a second edge? *4*

b) 1. Complete the table of values on the right, which associates the total number of edges in a graph with the sum of the degrees of all the vertices.

Number of edges	1	2	3	4	5	...	n
Sum of degrees	*2*	*4*	*6*	*8*	*10*	*...*	*2n*

2. What is the sum of the degrees of all the vertices of a graph with n edges? *2n*

c) 1. What is the sum of the degrees of all the vertices of a graph with 18 vertices? ***36***

2. How many edges are there in a graph for which the sum of the degrees of the vertices is equal to 52? ***26***

d) 1. Explain why the sum of the degrees of the vertices of a graph is an even number.

The sum of the degrees is equal to 2n where n $\in \mathbb{N}$***,***

and 2n is an even number for all n $\in \mathbb{N}$***.***

2. Explain why it is impossible to have only one vertex with odd degree in a graph.

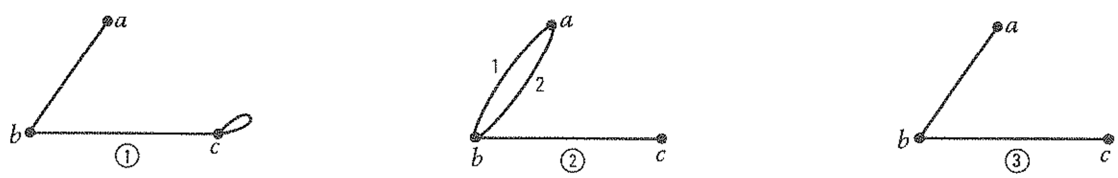
If only one vertex has odd degree, all of the other vertices therefore have even degree and the sum of the degrees of these other vertices is even. The total sum of the degrees of the vertices would then be odd (the sum of an even number with an odd number is odd) which would be a contradiction (see d) 1.).

3. Must the total number of vertices with odd degree in a graph be even or odd?

Even.

ACTIVITY 2 Simple graph

Consider the following graphs ① ② and ③.



a) Graph ① has a loop at vertex c. What is the sum of the degrees of the vertices of this graph?

6

b) Graph ② has 2 parallel edges. Indeed, vertices a and b are connected by two distinct edges written $a1b$ and $a2b$. What is the sum of the degrees of this graph? ***6***

c) Graph ③ does not have a loop. There are also no parallel edges in this graph. Such a graph is called simple. What is the sum of the degrees of this graph? ***4***

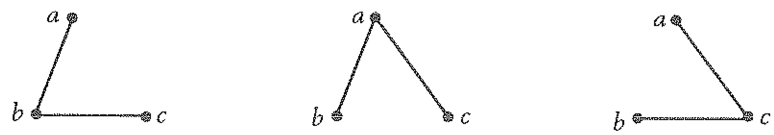
SIMPLE GRAPH

A graph is called simple when it does not have loops and when each pair of vertices is connected by at most one edge.

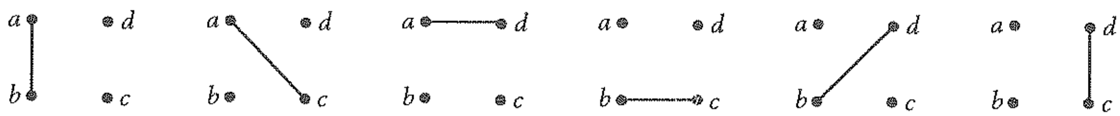
- Graph ① is not simple because it has a loop at vertex b.
- Graph ② is not simple because vertices a and b are connected by 2 edges.
- Graph ③ and ④ are simple.

6. Represent all possible simple graphs with

a) 3 vertices and 2 edges.



b) 4 vertices and 1 edge.

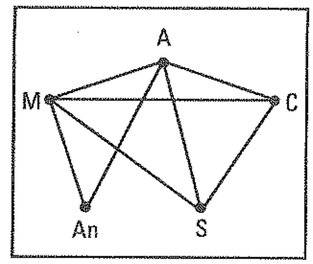


7. What is the maximal number of edges of a simple graph with

a) 2 vertices ***1 edge*** **b)** 3 vertices ***3 edges*** **c)** 4 vertices ***6 edges***

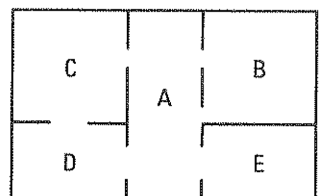
8. Allan has four friends: Muriel, Anna, Sarah and Celia. Sarah and Celia each have three friends. Sarah is friends with Celia but is not friends with Anna. Muriel is friends with each of Allan's friends.

a) Represent the situation using a graph.



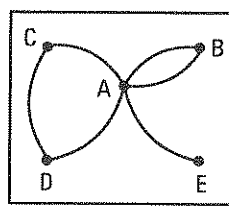
b) Who are Anna's friends? ***Allan and Muriel.***

9. The floor plan of an apartment is represented on the right.



A: Living room
B: Kitchen
C: Master bedroom
D: Bathroom
E: Bedroom

a) Represent the apartment using a graph where vertices correspond to the rooms of the apartment and edges correspond to the doors between rooms.



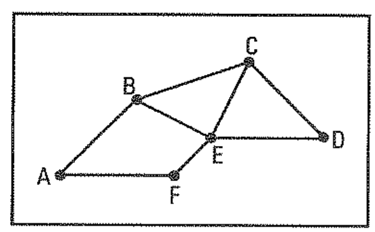
b) Identify the vertex with

1. the highest degree ***A***
2. the lowest degree ***E***

c) What is the sum of the degrees of the vertices of the graph? ***12***

ACTIVITY 3 Connected graph

The graph representing a region's road network is represented on the right. Each vertex of the graph corresponds to a village and each edge to a road connecting two villages.



a) Name a possible route between villages A and D.

Road AB followed by road BE followed by road ED.

b) Is it possible, starting from any village in this region, to reach any other village in the region?

Yes

If so, the graph is called connected.

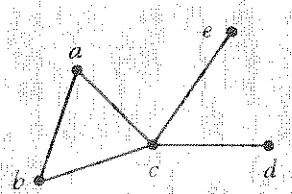
c) Does the graph remain connected if, because of a snowstorm,

1. the road is closed between villages A and B? *Yes*
2. the road is closed between villages A and B and villages E and F? *No*

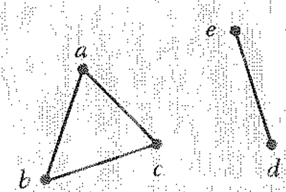
CONNECTED GRAPH

A graph is called **connected** if, between any two vertices of the graph, there exists an edge or a sequence of edges connecting these two vertices.

Ex.:



①



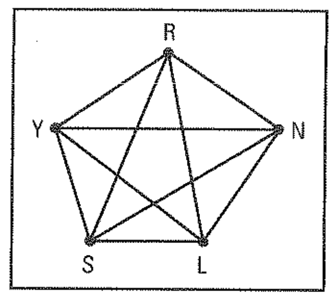
②

- Graph ① is connected.
- Graph ② is not connected because, for example, there is no edge or sequence of edges between vertices a and d.

ACTIVITY 4 Complete Graph

A travel agency wishes to install phone lines so that any two employees can communicate directly.

The agency has 5 employees: Richard, Yasmin, Simon, Lea and Nathalie.



a) Complete the graph on the right by representing each phone line by an edge.

b) Is the graph drawn connected? *Yes*

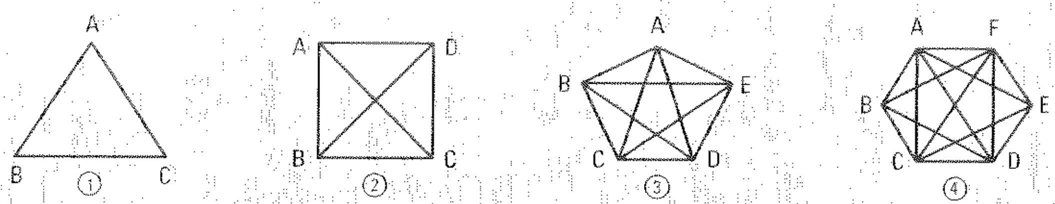
c) Are any two vertices of this graph adjacent? In other words, for any two vertices in the graph, does there exist an edge connecting these two vertices? *Yes*
If so, the graph is called **complete**.

d) 1. What is the degree of each vertex? *4*
2. How many edges are there in this graph? *10*

COMPLETE GRAPH

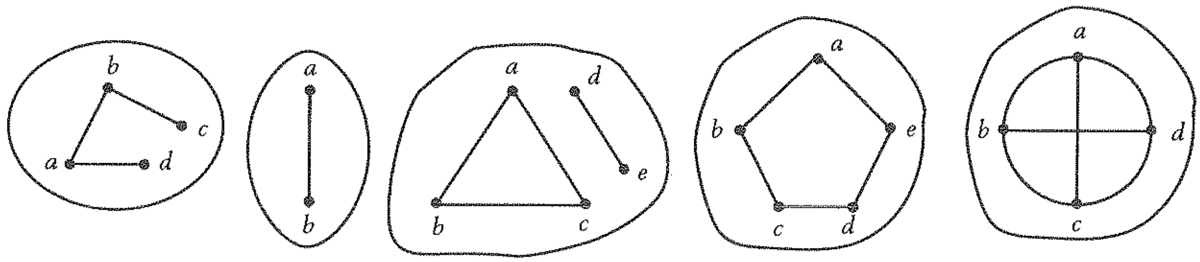
A graph is called complete if, between any two vertices, there exists an edge connecting these two vertices.

Ex.:



– Graphs ①, ② and ③ are complete.
– Graph ④ is not complete because edge BE does not exist.

10. Consider the following graphs.



G_1 G_2 G_3 G_4 G_5

a) Determine

1. the graphs which are connected. ***G_1, G_2, G_4 and G_5***
2. the graphs which are complete. ***G_2 and G_5***

b) Which are the graphs that are connected but not complete? ***G_1 and G_4***

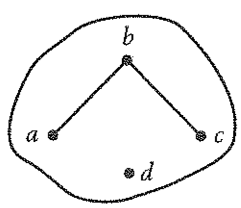
11. True or false?

a) If a graph is connected then it is complete. ***False***

b) If a graph is complete then it is connected. ***True***

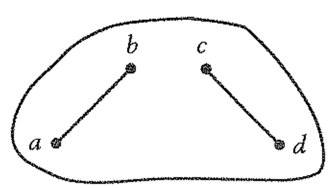
12. Find the missing edge in order to make the following graphs connected. Indicate all possible answers.

a)



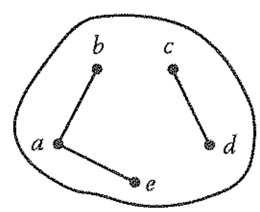
ad or cd or db.

b)



ac or ad or bc or bd.

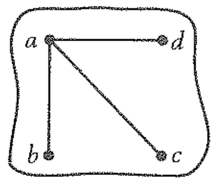
c)



ed or ec or ad or ac or bc or bd.

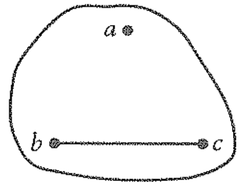
13. Find the missing edges to make the following graphs complete.

a)



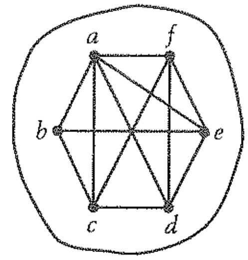
bc, bd and cd

b)



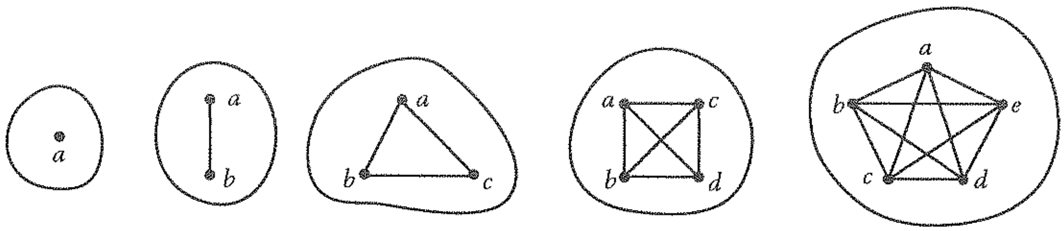
ab and ac.

c)



bf, bd and ce.

14. The following graphs are simple and complete.



a) What is the degree of each vertex of a complete graph of order

1. 3? ***2*** 2. 4? ***3*** 3. 5? ***4*** 4? n ***(n – 1)***

b) Complete the table of values giving the total number of edges of a complete graph according to the number of vertices.

Number of vertices	1	2	3	4	5	6
Number of edges	***0***	***1***	***3***	***6***	***10***	***15***

c) Verify that the total number of edges of a complete graph with n vertices is equal to $\frac{n(n-1)}{2}$.

d) What is the total number of edges of a complete graph with 10 vertices? ***45***

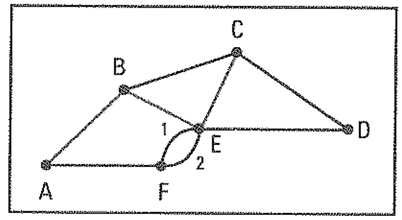
15. What is the minimum number of edges that a graph of order n must have in order to be complete?

(n – 1)

2.2 Chains

Activity 1 Chain – Simple chain

The road network of a region is illustrated on the right. Each vertex represents a village and each edge represents a road connecting two villages.



The red line in this graph is a chain.

This chain, which is a sequence of the three edges AB, BE and EC, is written: ABEC or CEBA.

Vertices A and C are called the endpoints of the chain.

The length of a chain is equal to the number of edges defining the chain.

a) What is the length of chain ABEC? ***3***

b) Find, in this graph, the shortest chain connecting the endpoints A and C and write this chain in two different ways. ***ABC or CBA***

c) The distance between two vertices is the length of the shortest chain joining these two vertices. What is the distance between

1. vertices A and C? ***2*** 2. vertices A and D? ***3***

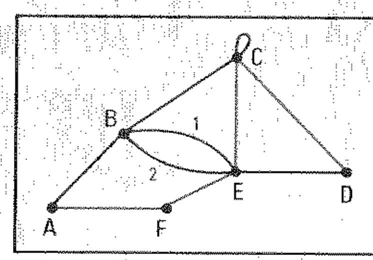
d) Find 3 different chains of length equal to 2 connecting vertices A and E. Write each chain in two different ways.

ABE or EBA; AF1E or E1FA; AF2E or E2FA

e) A chain is called simple if it does not have repeated edges. Indicate, among the following chains, the ones that are simple.

1. ABCEB ***Yes*** 2. ABCECD ***No*** 3. ABA ***No***

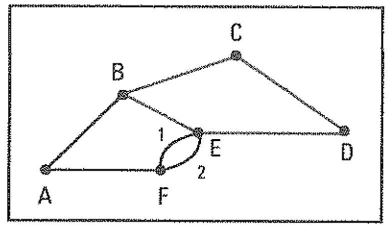
CHAIN – SIMPLE CHAIN



- A chain is a sequence of edges. The endpoints of the chain drawn in red on the right are A and D. It is written AFECD or DCEFA.
- The length of a chain is equal to the number of edges defining the chain. Chain AFECD has length 4.
- A chain can contain repeated vertices or edges. A chain is called simple if it does not contain repeated edges.
  - Chain AFECD is simple.
  - Chain AFECED is not simple because edge EC is repeated twice.
- The distance between two vertices A and B, written $d(A,B)$, is equal to the length of the shortest chain connecting these two vertices.
  Thus, we have: $d(A, B) = 1$, $d(A, C) = 2$, $d(A, D) = 3$.

ACTIVITY 2 Cycle – Simple cycle

Consider the graph on the right representing the road network of a region.



a) A chain beginning and ending at the same vertex is called cycle.

1. Is chain BCDEB a cycle? Justify your answer.
   Yes, because it begins and ends at the same vertex B.
2. Is chain BCDE a cycle? Justify your answer. ***No, because the endpoints of the chain are different.***

b) A cycle is called **simple** if it does not pass through the same edge twice. Among the following cycles, indicate the ones that are simple. If not, explain why.

1. BCDEB ***Simple cycle.***
2. BCDEBCB ***Non-simple cycle since it passes three times through edge BC.***
3. BE1F2EDCB ***Simple cycle.***
4. ABEBA ***Non-simple cycle since it passes, for example, twice through edge AB and twice through edge BE.***
5. AFA ***Non-simple cycle since it passes twice through edge AF.***

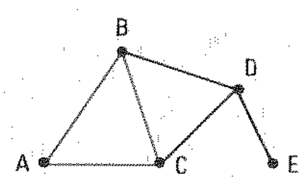
c) We can name a cycle by starting with any of its vertices. Name cycle BCDEB in seven other different ways.
BEDCB, CDEBC, CBEDC, DEBCD, DCBED, EBCDE, EDCBE.

d) Julian lives in village A. He leaves his village in the morning and strolls from village to village and returns home at night.

1. Explain why the path traveled by Julian is a cycle.
   The path traveled is a cycle beginning and ending at A.
2. Find all cycles of length 2 that Julian can travel.
   ABA and AFA.
3. Find the 10 different cycles of length 4 that Julian can travel.
   ABEBA, ABCBA, AF1E1FA, AF1E2FA, AF2E1FA, AF2E2FA, AF1EBA, AF2EBA, ABABA, AFAFA.
4. What are the only simple cycles of length 4 that Julian can travel?
   AF1EBA and AF2EBA

CYCLE – SIMPLE CYCLE

- A chain beginning and ending at the same vertex is called a cycle.
  The chain represented in red is a cycle that can be named in 6 different ways: ABCA, BCAB, CABC, ACBA, CBAC and BACB.
  These 6 different ways of naming the chain define one and only one cycle, drawn in red.



- A cycle is simple if it does not pass many times through the same edge.
  – ABCA is a simple cycle.
  – ABDCBA is not a simple cycle since it passes twice through edge AB.

1. For each of the following chains,
1. find its length 2. indicate if it is simple or not.

a) *ab* 1. ***1*** 2. ***simple***

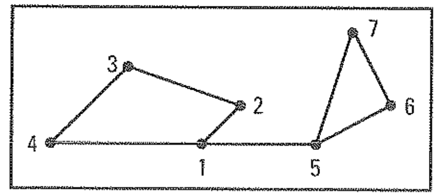
b) *abad* 1. ***3*** 2. ***not simple***

c) *abcdeba* 1. ***6*** 2. ***not simple***

d) *abcdb* 1. ***4*** 2. ***simple***

e) *abbcca* 1. ***5*** 2. ***simple***

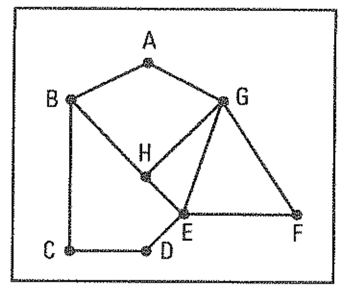
2. In the graph on the right, calculate the distance between the following vertices.



a) $d(3, 4)$: ***1*** b) $d(3, 5)$: ***3***

c) $d(7, 2)$: ***3*** d) $d(3, 7)$: ***4***

3. Consider the graph on the right.



a) For each of the following chains, indicate if the chain is simple and then calculate its length.
1. ABHED ***Simple; length: 4***
2. ABHGFED ***Simple; length: 6***
3. ABHGABC ***Not simple; length: 6***
4. HGEHBAGE ***Not simple; length: 7***

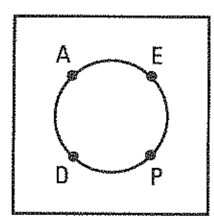
b) Calculate
1. $d(A, E)$ ***2*** 2. $d(A, D)$ ***3***

c) Indicate whether each of the following chains is a cycle.
If not, explain why.
1. ABHG ***No; because the chain does not end with vertex A.***
2. HEFGH ***Yes***
3. HEGHEH ***Yes***

d) Are the following cycles simple?
1. AGFEGA ***No*** 2. ABCDEGA ***Yes*** 3. GEFGHEG ***No***

e) Find, in this graph, a simple cycle of length equal to
1. 3 ***GEFG*** 2. 4 ***ABHGA*** 3. 5 ***BCDEHB*** 4. 6 ***ABHEFGA***
5. 7 ***ABCDEFGA*** 6. 8 ***HBCDEFGEH*** 7. 10 ***ABCDEHGFEGA***

4. Anthony, Eric, Philip and Dimitri are having breakfast sitting at a round table. The first person uses the sugar-bowl and then passes it to his immediate neighbour. The sugar-bowl goes around the other three people and then comes back to the first person.



Knowing that the path followed by the sugar-bowl is a simple cycle, find the 8 different ways of describing the only simple cycle of order 4 in this graph.

AEPDA, ADPEA, EPDAE, EADPE, PDAEP, PEADP, DAEPD, DPEAD

5. True or false?

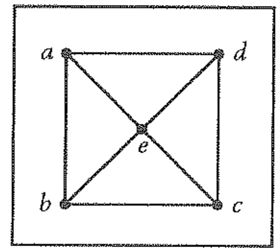
a) Any chain can be written in only two different ways. ***True***

b) Any cycle of length m can be written in $2m$ different ways. ***True***

c) A cycle is a chain beginning and ending at the same vertex. ***True***

d) A non-simple cycle can contain an infinite number of edges. ***True***

6. Consider the graph on the right.

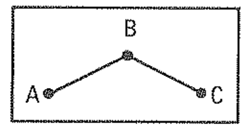


a) Find all simple cycles of length equal to 3.
abea, bceb, cdec, daed

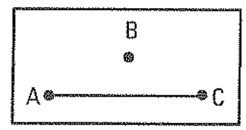
b) Find all simple cycles of length equal to 4 containing edge ab.
abeda, abcea, abcda

7. Construct a graph

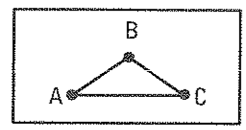
a) of order 3 containing a simple chain of length 2 and two vertices of degree 1.



b) of order 3 containing a chain of length 1 and which is not connected.

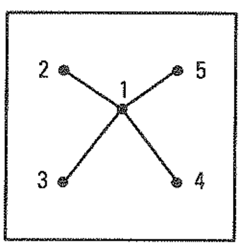


c) of order 3 containing a simple cycle.

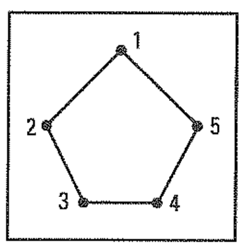


8. Construct a graph of order 5 which does not contain a cycle of order 3 and which contains a total of

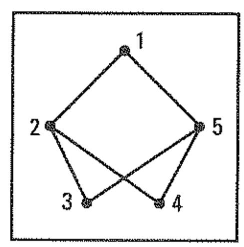
a) 4 edges.



b) 5 edges.

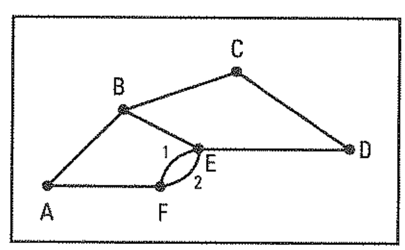


c) 6 edges.



ACTIVITY 3 Eulerian chain – Eulerian cycle

Consider the graph of the right representing the road network of activity 2.



a) Explain why the graph is connected.
Between any two vertices, there exists an edge or a sequence of edges connecting these 2 vertices.

b) The municipal services of the region have only one snowplough to remove snow on the road network. In order to use this snowplough efficiently, find a chain that allows the snowplough to pass through each road in the region once and only once.

Explain your solution with a chain.

For example, BAF1EBCDE2F.

c) Verify that the village at the beginning and the village at the end in the path found in b) are represented in the graph by odd degree vertices. ***deg(B) = 3; deg(F) = 3***

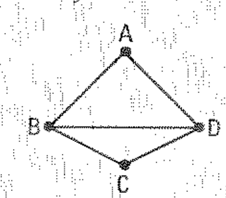
d) Is it possible to find in this road network a path for the snowplough which passes through each road once and only once and which begins and ends in the same village?

No

EULERIAN CHAIN – EULERIAN CYCLE

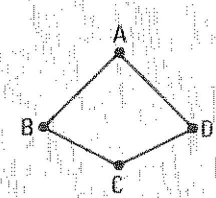
- In a connected graph, an Eulerian chain is a chain that passes through all edges of the graph once and only once. When an Eulerian chain begins and ends at the same vertex, it is called Eulerian cycle.
- There exists an Eulerian chain when the graph contains exactly two odd degree vertices. The odd degree vertices are then the beginning vertex and the ending vertex of the Eulerian chain.
- There exists an Eulerian cycle when all the vertices of the graph have even degree. The cycle can then begin at any vertex and end at this vertex.

Ex.:



BADCBD is an Eulerian chain.
deg(B) = 3; deg(D) = 3

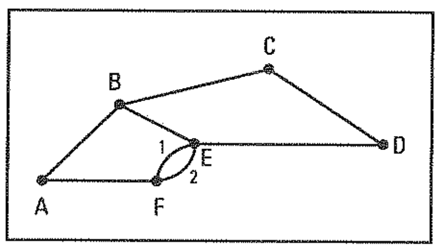
Ex.:



ABCDA is an Eulerian cycle.
All vertices have even degree.

ACTIVITY 4 Hamiltonian chain – Hamiltonian cycle

Consider the connected graph on the right representing the road network of activity 2.



Mr. Smith is a mailman and must deliver mail to all the villages of the region. He seeks an efficient route allowing him to pass through each village once and only once.

a) Find all possible routes if Mr. Smith wishes to begin in village A and end in village B.

AF1EDCB or AF2EDCB

b) Find a path starting in village A and returning to the starting village.

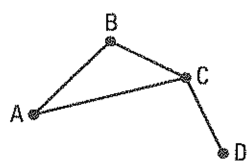
ABCDE1FA or ABCDE2FA

HAMILTONIAN CHAIN – HAMILTONIAN CYCLE

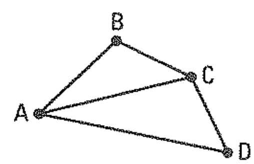
- In a connected graph, a Hamiltonian chain is a chain passing through all the vertices in the graph once and only once.

  When a Hamiltonian chain begins and ends at the same vertex, it is called Hamiltonian cycle.
- A sufficient but not necessary condition for a connected graph having n vertices ($n \geqslant 2$) to contain a Hamiltonian cycle is that each vertex of the graph is connected to at least half the other vertices, in other words, each vertex has degree greater than or equal to $\frac{n}{2}$.
- Note that a graph with a vertex of degree 1 does not contain a Hamiltonian cycle.

Ex.: The following graphs have 4 vertices. It suffices that all vertices have degree greater than or equal to 2 to have a Hamiltonian cycle.



- ABCD is a Hamiltonian chain.
- Since deg(D) = 1, there is no Hamiltonian cycle in this graph.



- ABCD is a Hamiltonian chain.
- All vertices have degree at least 2, so there exists a Hamiltonian cycle, namely ABCDA.

9. What is the difference between an Eulerian chain and a Hamiltonian chain?

An Eulerian chain passes once and only once through each edge of the graph while a Hamiltonian chain passes once and only once through each vertex of the graph.

10. What is the difference between

a) an Eulerian chain and an Eulerian cycle?

The Eulerian cycle returns to the starting vertex, the Eulerian chain, not necessarily.

b) a Hamiltonian chain and a Hamiltonian cycle?

The Hamiltonian cycle returns to the starting vertex, the Hamiltonian chain, not necessarily.

11. True or false?

a) In an Eulerian chain, a vertex can be repeated more than once. ***True***

b) In a Hamiltonian chain, an edge can be repeated more than once. ***False***

c) A non connected graph can contain an Eulerian cycle. ***False***

d) A non connected graph can contain a Hamiltonian cycle. ***False***

e) A graph with a vertex of degree 1 cannot contain a Hamiltonian cycle. ***True***

12. True or false?

a) A connected graph having 2 vertices with odd degree contains an Eulerian chain. ***True***

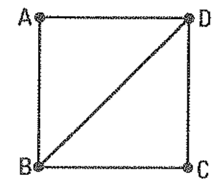
b) A connected graph whose vertices all have even degree contains an Eulerian cycle. ***True***

c) A graph with a vertex of degree 1 can contain an Eulerian cycle. ***False***

d) A non connected graph can contain an Eulerian chain. ***False***

e) A graph with 6 vertices contains a Hamiltonian cycle if each vertex touches at least 3 edges. ***True***

13. Consider the graph on the right.



a) Is the graph connected? ***Yes***

b) Calculate the degree of each vertex of the graph.
deg(A) = 2; deg(B) = 3; deg(C) = 2; deg(D) = 3.

c) Explain why this graph does not contain an Eulerian cycle.
The condition «all vertices must have even degree» is not satisfied.

d) 1. Explain why this graph contains an Eulerian chain.
There exist only two vertices with odd degree.

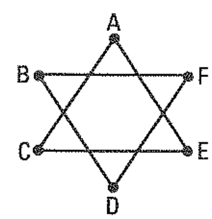
2. Which are the vertices at the endpoints of the Eulerian chain? ***B and D***

3. Find an Eulerian chain. ***BCDBAD***

e) 1. Explain why this graph contains a Hamiltonian cycle.
The graph is connected. Each vertex of the graph has degree greater than or equal to 2.

2. Find a Hamiltonian cycle. ***ADBC***

14. Consider the graph on the right.

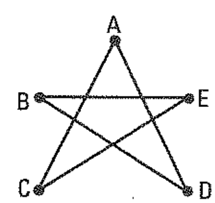


a) Is the graph connected? ***No***

b) Explain why the graph does not contain

1. an Eulerian cycle. ***The graph is not connected.***
2. a Hamiltonian cycle. ***The graph is not connected.***

15. Consider the graph on the right.



a) Is the graph connected? ***Yes***

b) Calculate the degree of each vertex.
deg(A) = 2; deg(B) = 2; deg(C) = 2; deg(D) = 2; deg(E) = 2.

c) 1. Explain why this graph contains an Eulerian cycle.
All vertices have even degree.

2. Find the Eulerian cycle contained in this graph. ***ACEBDA***

3. How many different ways are there to name this Eulerian cycle? ***10 ways.***

d) Is the Eulerian cycle obtained in c) also Hamiltonian?
Justify your answer. ***Yes, this cycle passes through each vertex once and only once.***

e) The sufficient condition for a connected graph with n vertices to contain a Hamiltonian cycle is the following:

«Each vertex of the graph must have degree greater than or equal to $\frac{n}{2}$».

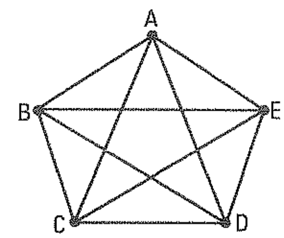
This condition is sufficient but not necessary.

Explain why this condition is not necessary.

In this graph, there are 5 vertices. $n = 5$ and $\frac{n}{2} = 2.5$.

The condition: each vertex must have degree greater than or equal to 2.5 is not satisfied but nonetheless the graph contains a Hamiltonian cycle.

16. Consider the graph on the right.



a) Is the graph connected? ***Yes***

b) Calculate the degree of each vertex.

deg(A) = 4; deg(B) = 4; deg(C) = 4; deg(D) = 4; deg(E) = 4.

c) 1. Explain why the graph contains an Eulerian cycle.

All vertices have even degree.

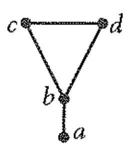
2. Find an Eulerian cycle. ***ABCDEBDACEA***

d) 1. Explain why the graph contains a Hamiltonian cycle.

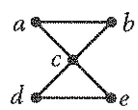
All vertices have degree greater than 2.5.

2. Find a Hamiltonian cycle. ***ABCDEA***

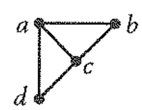
17. For each of the following graphs, indicate whether it contains an Eulerian cycle or else an Eulerian chain and then name the Eulerian cycle or chain.

a) 

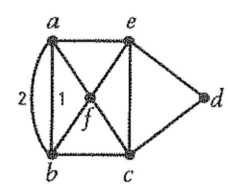
Cycle: no
Chain: abcdb

b) 

Cycle: yes
abcdeca

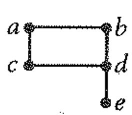
c) 

Cycle: no
Chain: abcdac

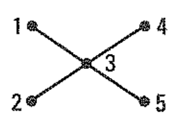
d) 

Cycle: yes
dea2bcfa1bfecd.

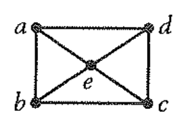
18. For each of the following graphs, indicate whether it contains a Hamiltonian cycle or else a Hamiltonian chain and then name the Hamiltonian cycle or chain.

a) 

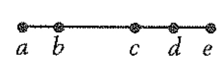
Cycle: no
Chain: bacde

b) 

Cycle: no
Chain: no

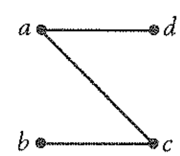
c) 

Hamiltonian cycle: yes
abceda

d) 

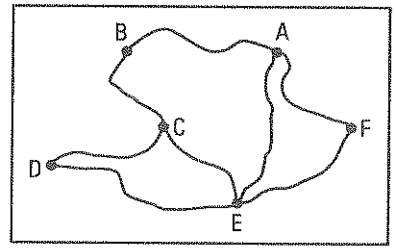
Cycle: no
Chain: abcde

19. Which edge must be added to this graph so that it contains an Eulerian cycle and a Hamiltonian cycle? ***Edge bd***



20. The graph on the right shows the cross-country skiing trails in a park of the Laurentians region. These trails connect cabins represented by the vertices A, B, C, D, E and F.

The person responsible for promoting the park wishes to offer a package in which a skier can start from any cabin and travel through all the trails once and only once and then return to the starting point at the end of the course.



a) Explain why it is impossible to offer such a package.

The package corresponds to an Eulerian cycle. It is impossible to find an Eulerian cycle because deg(C) = 3 (odd) and deg(A) = 3 (odd).

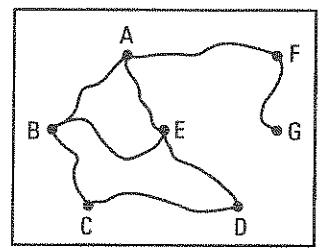
b) Explain why it is possible to travel along each trail once and only once without returning at the end of the course and then indicate the course.

There exists an Eulerian chain because there are two vertices with odd degree in this graph, namely A and C. The Eulerian chain is ABCDEFAEC.

c) Which trail must be built in the park in order to be able to return to the starting point?

The trail from A to C.

21. A travel agency wishes to offer tourists the option to visit all the popular sites in a town. These sites are represented on the graph on the right by the vertices A, B, C, D, E, F and G.



a) Explain why it is impossible to find an itinerary that passes through each site once and only once and returns to the starting point at the end of the itinerary.

We seek a Hamiltonian cycle. Such a cycle does not exist because deg(G) = 1.

b) Add one path (edge) joining two sites that the agency should consider in order to visit each site once and only once, then return to the starting point and then indicate an itinerary starting at site A.

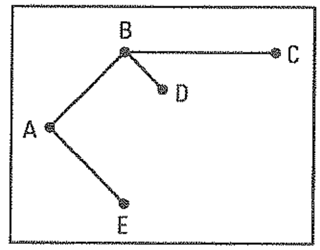
We add an edge connecting, for example, vertices E and G.

A Hamiltonian cycle would be ABCDEGFA.

2.3 Various graphs

Activity 1 Tree

The graph on the right illustrates 5 cabins located in a park. These cabins are represented by the vertices A, B, C, D and E. The promoter of the park would like that any two cabins be connected by a bike path (edge) or by a sequence of paths.



Each path connecting two cabins costing $5000, the promoter wishes to build a minimal number of paths in order to minimize costs.

a) Find a satisfactory solution for the promoter by adding edges to the graph.

b) Find other possible graphs. ***Various answers.***

c) Verify that each graph found in b) possesses the following characteristics:

1. The graph is connected.
2. The graph does not contain a simple cycle.
3. The graph has 4 edges.

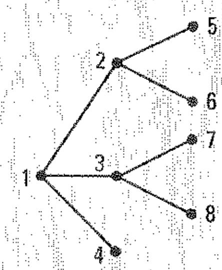
d) How many edges would the graph have if there were, in the park,

1. 6 cabins? ***5 edges*** 2. 7 cabins? ***6 edges*** 3. n cabins? ***(n − 1) edges.***

TREE

- A tree is a connected graph that does not contain a simple cycle.
- Any tree with n vertices has $(n - 1)$ edges.

Ex.: – The graph on the right is a tree.
– It has 8 vertices and 7 edges.



1. **a)** Is it possible to add an edge to a tree so that the graph obtained remains a tree? Explain why.

No, the graph we obtain will contain a simple cycle.

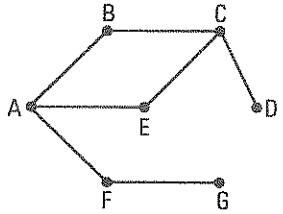
b) Is it possible to remove an edge from a tree so that the graph we obtain remains a tree? Explain why.

No, the graph we obtain will not be connected.

2. A tree has 16 vertices. How many edges does it have? ***15 edges.***

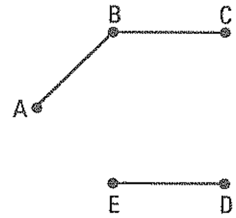
3. For each of the following graphs, add or remove an edge in order to obtain tree.

a)



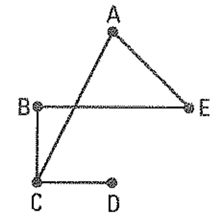
We remove edge AE, for example.

b)



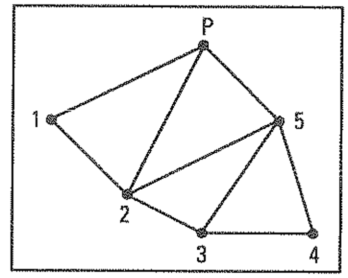
We must add an edge, for example edge AE.

c)



We remove edge AC, for example.

4. A watering system is represented on the graph on the right. The pump is located at vertex P and the five sprinklers are represented by the vertices 1 through 5. Edges correspond to pipes connecting sprinklers or connecting the pump to some of the sprinklers.



a) Indicate the maximal number of pipes that can be eliminated while keeping the watering system functional and name the pipes that we could eliminate.

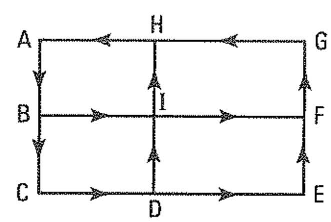
4 pipes; (P,2), (2,5), (3,5), (3,4)

b) Explain why the graph obtained in a), after eliminating the pipes, is a tree.

The graph we obtain does not contain a simple cycle and it is connected.

ACTIVITY 2 Directed graph

The graph on the right illustrates a neighbourhood in a city.
Each edge represents a street and the arrow on each edge indicates the direction of the traffic on the street.
The direction of the arrow on the edge connecting vertices A and B indicates that we can go from vertex A to vertex B but not the reverse. This directed edge is written (A, B) or simply AB.



a) Does edge (B, A) or BA exist? ***No***

b) ABIHA is a directed simple cycle since it is possible to successively connect vertices A, B, I, H and then return to vertex A.

Explain why no simple directed cycle exists between vertices B, C, D and I.

The direction of traffic does not allow us to start from vertex B and return to vertex B by going through vertices C, D and I.

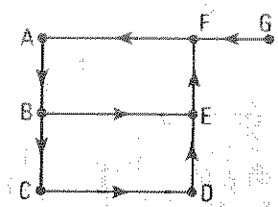
c) Name all directed simple cycles found in this graph.

ABIHA, ABCDIHA, ABIFGHA, ABCDEFGHA, ABCDIFGHA.

d) Does there exist a directed Hamiltonian chain in this graph?
If so, name it. ***Yes, the chain IHABCDEFG or EFGHABCDI.***

DIRECTED GRAPH

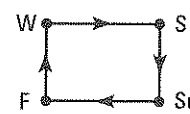
- A directed graph is a graph in which each edge is oriented. On each edge connecting two vertices, an arrow indicates the direction of motion between the two vertices.



- In a directed graph, it is usual to call a directed edge an arrow, to call a directed chain a path and to call a directed cycle a circuit. We will not use these terms in order to simplify the language.

  Ex.: In the directed graph on the right,
  - edge AB exists and edge BA does not exist.
  - the distance between vertices F and E is equal to 3 because the shortest chain FABE connecting vertices F and E contains 3 edges.
  - the distance between vertices F and G does not exist because there is no chain connecting vertices F and G.
  - there exist only two directed simple cycles: ABEFA and ABCDEFA.
  - there exists only one directed Hamiltonian chain: GFABCDE.
  - there does not exist any directed Eulerian cycle nor directed Hamiltonian cycle.

5. a) Represent the four seasons winter (W), spring (S), summer (Su) and fall (F) using a directed graph.



b) 1. Do we observe a directed Eulerian cycle? ***Yes, WSSuFW.***

2. If so, how many different ways are there to name the season cycle? ***4***

c) 1. Do we observe a directed Hamiltonian cycle? ***Yes, WSSuFW.***

2. If so, how many different ways are there to name it? ***4***

6. Five friends took a mathematics exam.
- David has a better score than Valerie.
- Nathalie has a better score than Eric.
- Karen has a better score than David.
- Valerie has a better score than Nathalie.

a) Construct a directed graph representing this situation. (Explain how you oriented each edge.)

The arrow is oriented from K to D on edge KD to indicate that K has a better score than D.

b) Does there exist a directed Hamiltonian chain. If so, name it. ***Yes, KDVNE.***

c) Which of the five friends has

1. the lowest score? ***Eric*** 2. the best score? ***Karen***

7. How many different ways are there to name, in a directed graph,

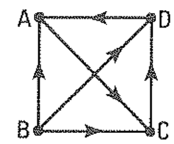
a) a directed Hamiltonian chain? ***Only one way.***

b) a directed simple cycle joining n vertices? ***n ways.***

8. Fours teams A, B, C and D compete in a soccer tournament. The tournament rules require that each team plays against each of the other three. The winning team is the one which has the most wins. The table on the right gives, for a match (X, Y) where team X plays team Y, the result (x, y) where x represents the number of goals scored by team X and y the number of goals scored by team Y.

(X, Y)	(x, y)
(A, B)	(2, 3)
(A, C)	(3, 1)
(A, D)	(1, 3)
(B, C)	(1, 0)
(B, D)	(3, 2)
(C, D)	(2, 1)

a) Construct a directed graph where edge XY indicates that X won over Y.

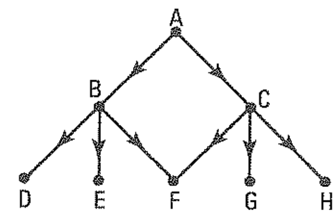


b) Which team wins the tournament? ***B***

c) For each of the following questions, justify your answer if your answer is affirmative. Do we observe in this graph

1. a directed Eulerian cycle? ***No***
2. a directed Eulerian chain? ***No***
3. a directed Hamiltonian cycle? ***No***
4. a directed Hamiltonian chain? ***Yes, BCDA***

9. The graph on the left shows the organizational chart of a company. In this directed graph, an edge XY indicates that Y works under X. In this company, there is a president, two vice-presidents and five service managers in charge of 24 employees each. Each employee works under only one service manager.



a) Name

1. the president ***A***
2. the two vice-presidents. ***B and C***

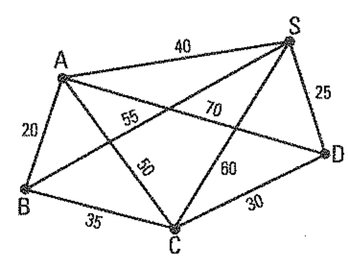
b) Find the only manager working under the two vice-presidents. ***F***

c) How many people work under a vice-president? ***75***

ACTIVITY 3 Weighted graph

Rafael must deliver mail to 4 customers.

On the graph on the right, the mail sorting centre is represented by vertex S and the four customers are represented by vertices A, B, C and D.



The edges of the graph indicate the various possible routes between the sorting centre and each customer or possible routes between customers. The number associated with each edge represents the average travel time (in minutes) for the route corresponding to this edge.

Rafael wishes to minimize the duration of his route.

He leaves the sorting centre (S) and wants to return at the end of his route while visiting each customer once and only once.

a) Explain why the route he seeks is a directed Hamiltonian cycle. ***It starts from vertex S and must return to vertex S after visiting all the other vertices once and only once.***

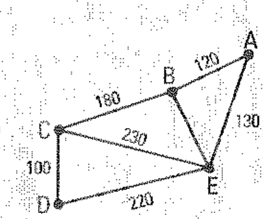
b) Find all possible directed Hamiltonian cycles as well as the traveling time for each cycle.
SABCDS (150), SADCBS (230), SBACDS (180), SBCADS (235), SCDABS (235), SCBADS (210)

c) Which route must Rafael choose? ***SABCDS***

WEIGTHED GRAPH

- A weighted graph is a graph where there are numerical values on the edges. These values can represent a distance, a duration, a cost, etc.
- The value of a chain is equal to the sum of the values of the edges that constitute the chain.
- A weighted graph can be directed or non directed.

Ex.: The graph on the right indicates the distance (in km) between different towns connected by a railroad.
The value of the chain ABCD is equal to 400 km and corresponds to the total distance traveled when we start from town A and travel to town D passing successively through towns B and C.



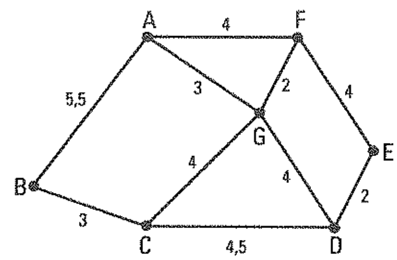
10. In the graph on the right,

a) calculate the value of chain BCGF. ***9***

b) calculate the length of chain BCGF. ***3***

c) calculate the distance between B and F. ***2***

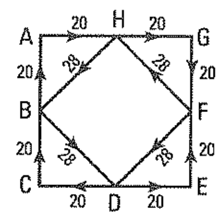
d) find a Hamiltonian cycle in this graph and calculate its value.
Various answers, for example BCDEFGAB with value 24.



11. Consider the graph on the right.

a) 1. Find the shortest chain from vertex A to vertex C and then calculate its value. ***ABC; value 40.***

2. Find the chain with the smallest value connecting vertices A and D and indicate its value.
ABD; value 48.



b) Find three Hamiltonian chains and calculate their value.
GFHABCDE with value 148.
ABCDEFGH with value 140.
AHBCDEFG with value 148.

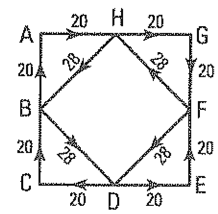
c) What is the only Hamiltonian cycle in this graph? What is its value?
ABCDEFGHA with value 160.

d) Is it possible to find a Hamiltonian chain with value less than 140 in this graph? ***No***

e) Find an Eulerian cycle in this graph and calculate its value.
HBDFHGFEDCBAH with value 272.

f) Explain why finding the minimal value of an Eulerian cycle does not make sense.
In an Eulerian cycle, all the edges in the graph are covered once and only once. Thus, the value of an Eulerian cycle in a graph remains constant.

12. The weighted graph of exercise 11 is now a directed graph.



a) Find the shortest chain from vertex A to vertex C and then calculate its value.

AHBDC; value: 96.

b) Is it possible to find

1. a directed Hamiltonian cycle? *No*
2. a directed Eulerian cycle? *Yes, AHGFDEFHBDCBA*

c) Find the six directed simple cycles in this graph and their value.

AHBA; HGFH; FDEF; CBDC. They all have value 68.

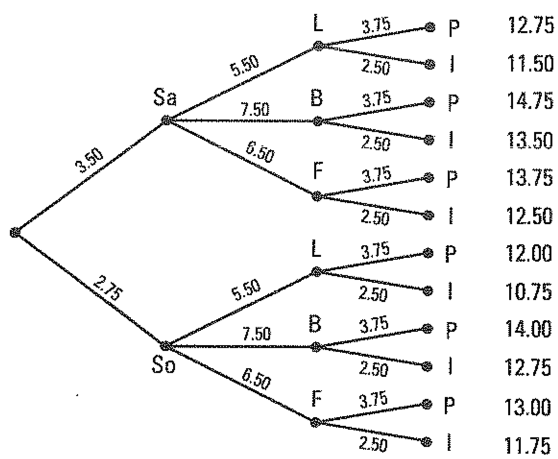
HBDEFH with value 124 and AHGFDCBA with value 148.

13. The restaurant «Delicious Food» offers at lunchtime a choice of 2 appetizers: salad ($3.50) or vegetable soup ($2.75), a choice of 3 main courses: lasagna with tomato sauce ($5.50), beef with mushrooms ($7.50) or fish and chips ($6.50) and a choice of 2 desserts: apple pie ($3.75) or ice cream ($2.50).

A complete lunch menu includes an appetizer, a main course and a dessert.

a) How many possible menus are there? *12 menus.*

b) Construct a weighted tree illustrating the various possible menus.



c) What are the menus that Sylvia can afford if she has less than $12?

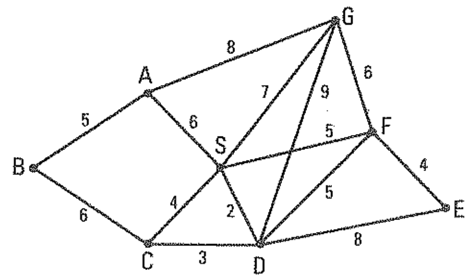
(Salad, lasagna, ice cream), (soup, lasagna, ice cream) or (soup, fish and chips, ice cream).

2.4 Optimization problems

Activity 1 Optimal chain - Optimal cycle

Every day, Alan delivers dairy products to 7 convenience stores, represented on the weighted graph on the right by vertices A, B, C, D, E, F and G.

The distribution centre is located at vertex S. The number on each edge is the distance (in km) between 2 customers.



a) Alan wishes to go from customer B to customer F.

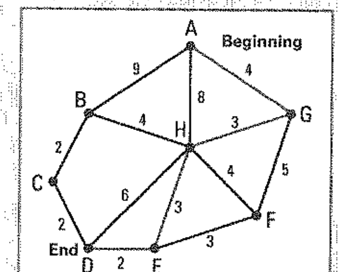
1. What is the shortest path? ***BCDF***
2. What is the shortest path if he must pass through the distribution centre? ***BCSF***

b) He leaves the distribution centre in the morning and must visit all the convenience stores once and only once during the day. Find the shortest route if

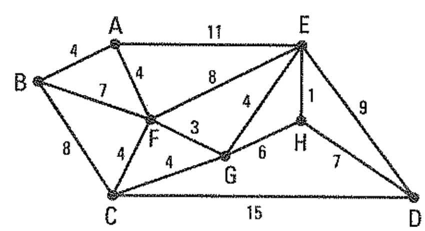
1. he does not end his day at the distribution centre. ***SDCBAGFE (34 km)***
2. he ends his day at customer D. ***SCBAGFED (41 km)***
3. he ends his day at the centre. ***SDEFGABCS or SCBAGFEDS (43 km)***

OPTIMAL CHAIN

In the graph on the left, the chain with minimal value connecting beginning vertex A and ending vertex D is the chain AGHED with value 12.

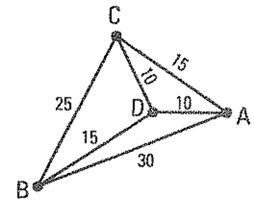


1. Consider the weighted graph on the right.
Find the chain with minimal value connecting vertices B and D.
The chain BFGEHD with value 22.



2. The average traveling time (in min) on Sundays between the houses of four friends Andy, Ben, Caroline and Diana is indicated on each edge.

It's Sunday and Andy leaves home at 9:00 in the morning to visit each of the three other friends. Knowing that he spends an hour at each of his friends', when will he be back home if he wants to return as soon as possible?



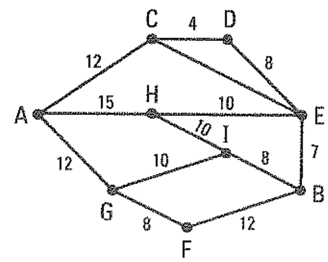
He travels the Hamiltonian cycle ACBDA or ADBCA with traveling time 65 minutes. He will be back home at 13:05.

3. The weighted graph on the right was made following a study on the construction of different possible road sections allowing sites A and B in a mountainous region to be connected by a road.

On each edge (road section), the cost (in millions of dollars) of the corresponding section is indicated.

Find the minimal cost chain connecting points A and B.

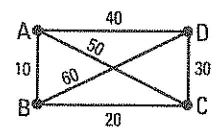
The chain AGIB with value 30 million dollars.



4. Find a Hamiltonian cycle starting at vertex A and having

a) minimal value. ***ABCDA or ADCBA with value 100***

b) maximal value. ***ACBDA or ADBCA with value 170.***

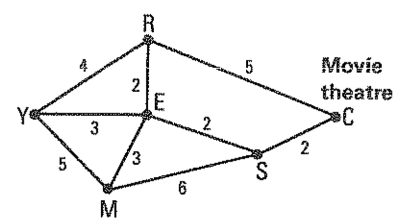


5. Elie leaves home by car and picks up his friends Richard, Yasmin, Morris and Sonia to go to the movies.

The number on each edge indicates the average traveling time (in min) from one house to another.

Find the shortest route.

ERYMESC with duration 18 minutes.

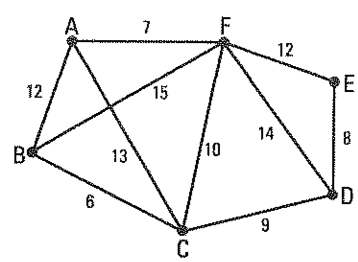


ACTIVITY 2 Optimal tree

The graph on the right was obtained following a study of the costs (in thousands of dollars) of building trails connecting six cottages in the Lanaudiere region.

a) What would be the total cost if we were to build all the trails in this graph?

$106 000



b) To control the costs, we wish to eliminate a few trails while making sure it is possible to reach any cottage from any other one.

1. Is it possible to eliminate some trails (edges) while making sure that any cottage can be reached from any other cottage? ***Yes***
2. In the choice of edges (trails) to keep, can we keep edges forming a simple cycle? ***No***
3. What property must the graph verify after eliminating unnecessary trails?
   The graph must remain connected.

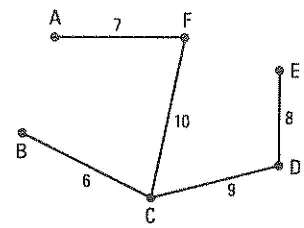
c) Find a procedure that allows us to connect any cottage starting from any other at minimal cost.

1. We choose the least expensive edge (trail).

2. We choose among the remaining edges the least expensive one that does not form a simple cycle.

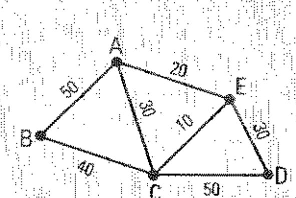
3. We repeat step 2 until we obtain a connected graph.

d) 1. Represent the graph that answers question c).
2. Verify that the graph you found is a tree.
3. What is the minimal cost? ***$40 000***



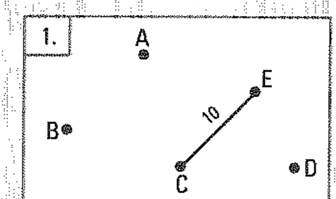
OPTIMAL TREE

Given a directed weighted graph, it is possible to find in this graph a weighted tree with minimal or maximal value.

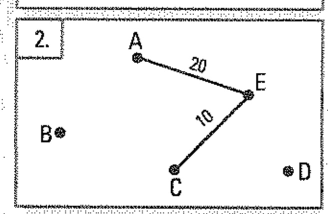


We use the following procedure to determine the minimal value tree.

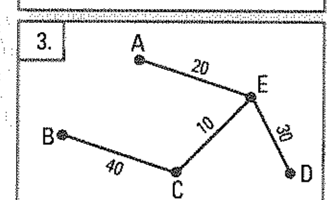
1. We draw the graph with only the vertices then we draw the edge with the lowest value.



2. We draw, as we go, the edge with the smallest value among the remaining edges while always avoiding adding an edge that forms a simple cycle.

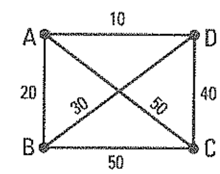


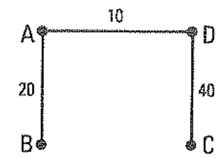
3. We continue this procedure until we obtain a tree.



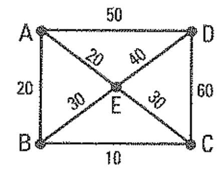
6. For each of the following graphs, find the minimal value tree.

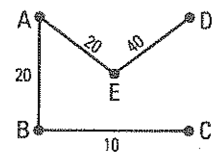
a)



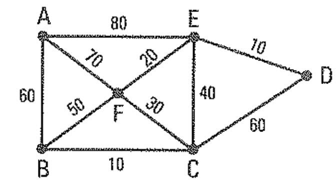


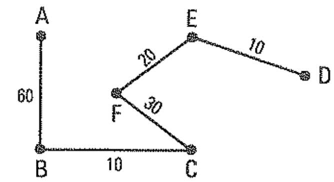
b)





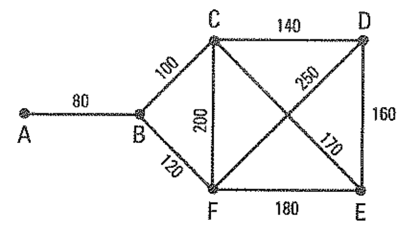
c)

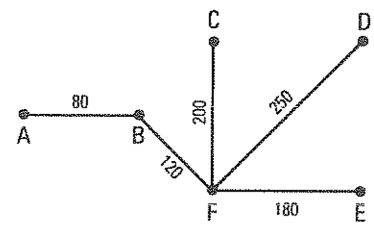




7. A road network connecting villages A, B, C, D, E and F is represented on the graph on the right. The value associated with each edge represents the average number of vehicles transiting each day on each road of the network.

After a snowstorm, municipal services decide, for their first snow removal operation, to clear the roads that are used the most and make sure that each village is accessible from any point of the network.





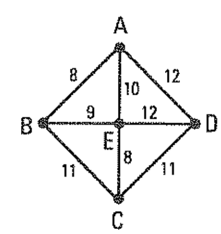
a) How many roads must be cleared? ***5 roads***

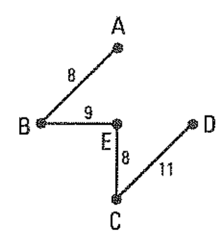
b) What are the roads that will not be cleared on this first snow removal operation?

BC, CD, DE and CE.

8. Roger wants to install a watering system in his garden. On the graph on the right, sprinklers are represented by vertices A, B, C, D and E and the pipes connecting the sprinklers are represented by the edges. The value associated with each edge represents the length (in m) of the various possible pipes.

Roger wishes to connect all the sprinklers while minimizing the total length of the pipes.



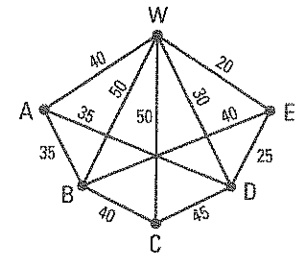


a) How many pipes must he keep? ***4 pipes***

b) What is the total length of the pipes Roger must install?

36 m

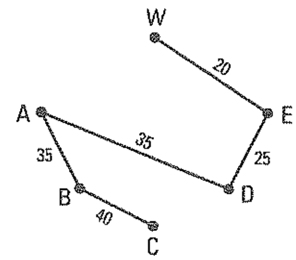
9. A real estate promoter wants to install a water supply system connecting five houses in a construction project. The graph on the right illustrates the well (W) and the five houses represented by vertices A, B, C, D, and E.



The possible conduits are represented by edges. The value associated with each edge corresponds to the installation cost (in hundreds of dollars) of each conduit.

a) How many conduits must we plan? ***5 conduits***

b) What is the minimal installation cost to connect the well to the five houses? ***$15 500***

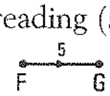


ACTIVITY 3 **Critical path**

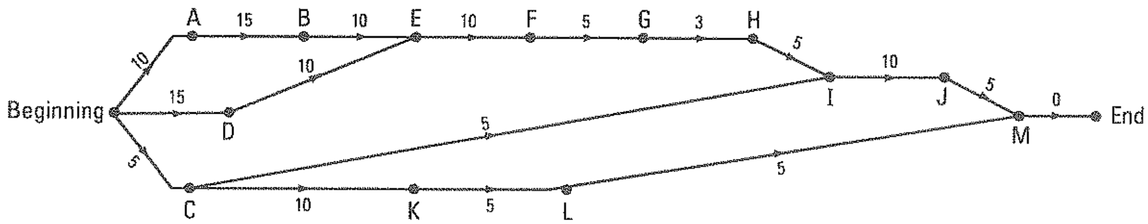
The production and distribution of a book comprises several steps between the receiving of the manuscript and the distribution of the book. Some steps can be carried out simultaneously, others can be carried out only if one or more other steps have already been completed. The following table indicates the different steps, their completion time (in days) and the prerequisite steps.

	Steps	Duration (days)	Prerequisite step(s)
A	Composition of texts	10	none
B	Computer input of texts	15	A
C	Design of cover page	5	none
D	Graphic design and figures	15	none
E	Layout of texts and figures	10	B and D
F	1st proof-reading	10	E
G	2nd proof-reading	5	F
H	3rd proof-reading	3	G
I	Collating of book's pages	5	C and H
J	Printing of book	10	I
K	Design and production of advertising material	10	C
L	Distribution of advertising material	5	K
M	Distribution of book	5	J and L

In the directed weighted graph below, each vertex represents a step.

The constraint: "1st proof-reading (step F) is prerequisite to 2nd proof-reading (step G)" is represented by the directed edge: 

The duration «5 days» of step G is indicated above edge FG.



Note that any directed edge ending at End such as M —0→ F has value 0.

We observe, in this graph, 4 paths (directed chains) connecting the Beginning and the End, which are: Beginning ABEFGHIJM End, Beginning DEFGHIJM End, Beginning CIJM End and Beginning CKLM End.

a) Verify that the directed chain Beginning ABEFGHIJM End is the one with maximal value in this graph.

Value of the chain: 73.

This directed chain is called **critical path**. Its value represents the necessary and sufficient time for the production of this book.

b) How much time does it take to produce this book? ***73 days***

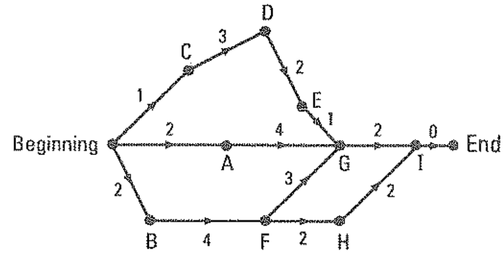
c) The graphic designer had to take a sick leave at the beginning of the production process. What is the consequence on the time necessary to complete the production of this book if the designer was absent for

1. 10 days ***No consequence***   2. 15 days ***The production time increases by 5 days.***

CRITICAL PATH

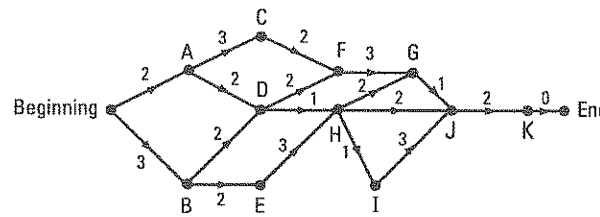
- A task requiring the completion of several steps can be represented by a directed weighted graph taking into account
  – the prerequisite steps,
  – the steps that can be carried out simultaneously.
- The critical path is the directed chain in the graph that has maximal value.
  The value of the critical path represents the minimum time for the completion of all the steps that constitute the task.
  Ex.: See activity 1.
  The critical path Beginning ABEFGHIJM End has value 73.
  73 days are necessary for the production and distribution of the book.

10. In each of the graphs on the right, find the critical path and give its value.



a) *Beginning BFGI End*

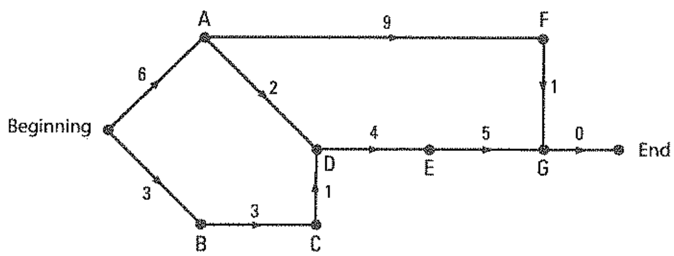
Value = 11



b) *Beginning BEHIJK End*

Value = 14

11. The graph below represents the various steps necessary for the completion of a task. The number associated with each edge represents the number of work days necessary to carry out the corresponding step. The cost of a work day comes to $2000.



a) 1. Find the critical path. *Beginning ADEG End*

2. What is the minimal total cost to complete this task?

Value of the critical path = 17 days; minimal cost: $34 000.

b) Because of a technical difficulty, step C required 2 additional days. What is the increase in the minimal total cost?

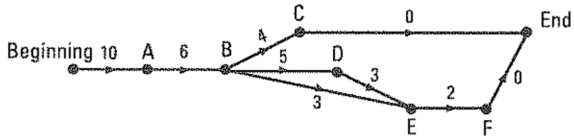
The new critical path Beginning BCDEG End has value 18 days. The minimal total cost increases by $2000.

12. The completion of a task comprises 6 steps denoted by A, B, C, ...,F. It is possible to carry out several steps at the same time.

The table on the right indicates, for each step, the completion time, in days, as well as the prerequisite steps.

Steps	Duration	Perequisite step(s)
A	10 days	None
B	6 days	A
C	4 days	B
D	5 days	B
E	3 days	B and D
F	2 days	E

a) What is the minimal time required to complete the 6 steps required by this task?



Critical path: Beginning ABDEF End.

The minimal time is 26 days.

b) What is the minimal time if step B takes one less day and step C takes 2 more days?

We gain one day.

13. Melina wants to prepare dinner. She estimates that the steps: set the table (T), prepare the salad (Sa) and prepare the dessert (D) each take 5 minutes.

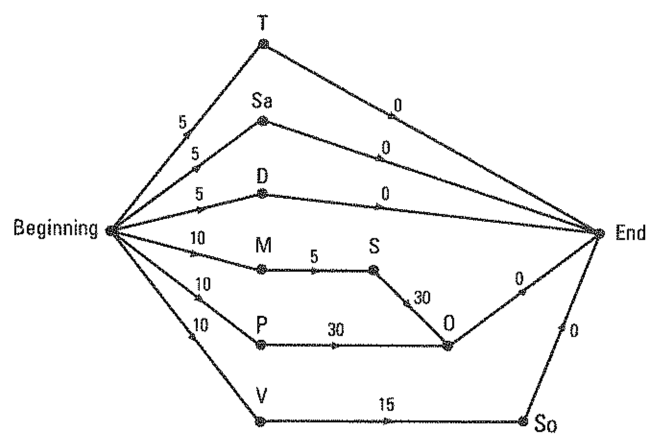
The preparation of the main course consists in thawing the meat (M), which takes 10 minutes, then seasoning it (S), which takes 5 minutes and can be done by only one person. She estimates that the step: peel the potatoes (P) takes 10 minutes and that the step: cook the meat and the potatoes in the oven (O) takes 30 minutes.

In addition, she wishes to prepare a soup that requires 10 minutes for peeling the vegetables (V) and 15 minutes to cook (So).

a) In a table, place the different steps, their duration (in min) and the prerequisite steps.

	Steps	Duration (min)	Prerequiste step(s)
T	*Set the table*	*5*	*None*
Sa	*Prepare salad*	*5*	*None*
D	*Prepare dessert*	*5*	*None*
M	*Thaw meat*	*10*	*None*
S	*Season meat*	*5*	*M*
P	*Peel potatoes*	*10*	*None*
O	*Cook meat and potatoes in oven*	*30*	*S and P*
V	*Peel vegetables*	*10*	*None*
So	*Cook soup*	*15*	*V*

b) Represent the preparation of the dinner using a graph.



c) What is the minimum time it will take Melina to prepare dinner? ***45 min.***

d) Melina asks her brother Adrian to help her prepare the dinner. Will Adrian's help reduce the minimum time to prepare the dinner? Justify your answer.

Adrian's help cannot reduce the duration of thawing (M) and the cooking time (O). The step: season the meat (S) cannot be reduced since the seasoning can only be done by one person. Adrian's help will not reduce the minimal time for the preparation of the dinner.

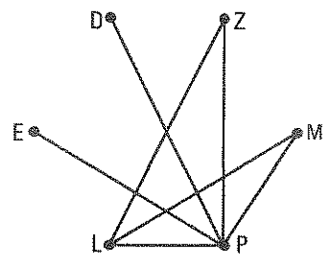
ACTIVITY 4 Graph coloring

The manager of a circus' parade must group dromedaries (D), elephants (E), lions (L), panthers (P), monkeys (M) and zebras (Z). Incompatibilities between the animals are listed in the table on the right.

Animal	Incompatibilities
D	P
E	P
L	P, M, Z
P	E, D, L, M, Z
M	L, P
Z	L, P

a) Draw a graph whose vertices are the various species of animals D, E, ..., Z and whose edges are the incompatibilities.

Thus, incompatibility between dromedaries and panthers will be represented by an edge DP.

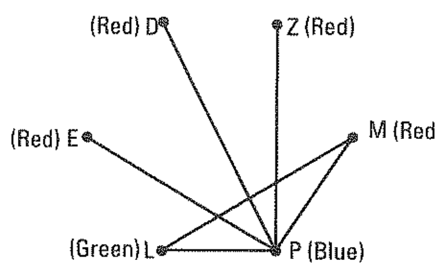


b) 1. Which is the vertex with highest degree? ***P***

2. Color the vertex with highest degree using a color of your choice.

c) Color the other vertices while respecting the following rules.

1. Two adjacent vertices must not have the same color.
2. A minimum number of different colors must be used to color this graph.



d) The number of distinct colors used to color this graph is called «chromatic number».

1. What is the chromatic number of this graph? ***3***
2. Interpret the chromatic number in this situation.

It represents the minimum number of animal groups that will march in the parade.

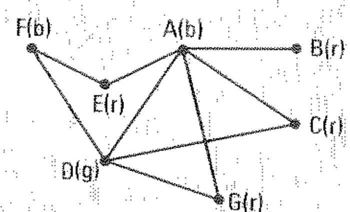
GRAPH COLORING

- When solving a problem, it can prove useful to color the vertices of a graph. The coloring rules are as follows:
  1. Two adjacent vertices always have different colors.
  2. A minimal number of different colors must be used to color the vertices of the graph.

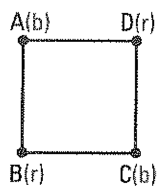
  In practice, after ordering the vertices in decreasing order of degree, we color the vertex with highest degree then we color, in order, the other vertices while respecting the two coloring rules. We therefore reuse, as soon as possible, the colors that have already been used.

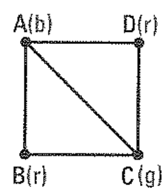
  Ex.: Some students in class are talkative. The teacher has made the graph below. Anita, Beatrix, Celia, Daniel, Evelyn, Frank and Georges are represented by the graph's vertices. He connects two vertices (students) in the graph with an edge to indicate that the two students must not sit next to each other.

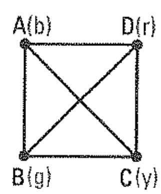
- The chromatic number of the graph is 3.
  The teacher can seat Georges, Celia, Beatrix and Evelyn together, Anita and Frank together while Daniel will be isolated from the other students mentioned.

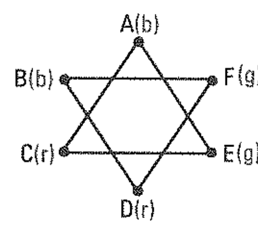


14. Find the chromatic number of the following graphs.

a)


b)


c)


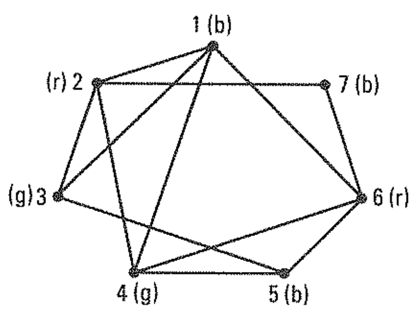
d)


Legend: blue (b); red (r); green (g); yellow (y)

a) *2* b) *3* c) *4* d) *3*

15. The person responsible for exams in a school must plan the exam schedule taking into account scheduling conflicts. There is a scheduling conflict when two exams take place at the same time and there exists at least one student in this school that must pass these two exams. Determine how many scheduling blocks are necessary so that each student in this school can pass the exams, then define the scheduling blocks.

Exams	Incompatibilities
1	2, 3, 4 and 6
2	1, 3, 4 and 7
3	1, 2 and 5
4	1, 2, 5 and 6
5	3, 4 and 6
6	4, 5 and 7
7	2 and 6



Chromatic number = 3.

There will be 3 scheduling blocks.

Blue scheduling block: exams 1, 5 and 7.

Red scheduling block: exams 2 and 6.

Green scheduling block: exams 3 and 4.

Legend: *blue (b)*
red (r)
green (g)

Evaluation 2

1. Complete the following definitions.

a) A graph is connected if *between any 2 vertices, there exists an edge or a sequence of edges connecting these 2 vertices.*

b) A graph is complete if *between any 2 vertices, there exists and edge connecting these 2 vertices.*

c) A chain is *a sequence of consecutive edges.*

d) A simple chain is *a sequence of consecutive edges without any repeated edges.*

e) A cycle is *a sequence of consecutive edges beginning and ending at the same vertex.*

2. Complete the following definitions.

a) The order of a graph is *the number of vertices in this graph.*

b) The degree of a vertex is *equal to the number of edges that reach that vertex.*

c) Two vertices are adjacent when *there exists an edge connecting these 2 vertices.*

d) The length of a chain is *equal to the number of edges in the chain.*

e) The distance between two vertices is *equal to the length of the shortest chain connecting these 2 vertices.*

3. Complete the following definitions.

a) A chain is Eulerian if *it passes through every edge in the graph once and only once.*

b) A chain is Hamiltonian if *it passes through every vertex in the graph once and only once.*

c) An Eulerian cycle is *an Eulerian chain beginning and ending at the same vertex.*

d) A Hamiltonian cycle is *a Hamiltonian chain beginning and ending at the same vertex.*

e) A tree is a graph that is *connected and without any simple cycle.*

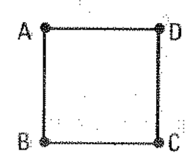
f) The value of a chain is *equal to the sum of the values of the edges constituting the chain.*

4. True or false?

a) The graph on the right is connected. *True*

b) The graph on the right is complete. *False*

c) ABCB is a simple chain. *False*

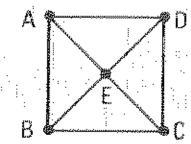


d) ABCBA is a non-simple cycle. ***True***

e) ABCDA is an Eulerian cycle. ***True***

f) ABCDA is a Hamiltonian cycle. ***True***

5. Consider the graph on the right.

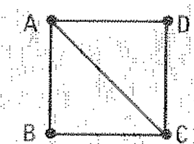


a) Does there exist an Eulerian cycle in this graph? Justify your answer.
No, since the vertices do not have even degree.

b) Does there exist a Hamiltonian cycle in the graph? If so, justify your answer and name a Hamiltonian cycle.
Yes, the graph has 5 vertices (n = 5). Each vertex has degree greater than $\frac{n}{2}$. ABECDA is a Hamiltonian cycle.

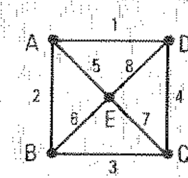
c) Does there exist an Eulerian chain in this graph? Justify your answer.
No, in this graph there exists 4 vertices with odd degree (A, B, C and D). The existence condition for an Eulerian chain is that the graph must have at most 0 or 2 vertices with odd degree.

6. Consider the graph on the right.
Explain why there exists an Eulerian chain and then find the Eulerian chains in this graph.

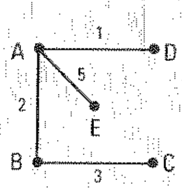


There exists exactly 2 vertices with odd degree. (A and C have degree 3). ABCADC, ADCABC, ABCDAC and ADCBAC.

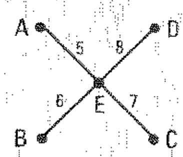
7. In the graph on the right, find the tree having



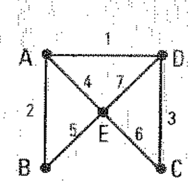
a) the smallest value



b) the greatest value



8. Consider the graph on the right.



a) Does this graph have an Eulerian cycle. ***No***

b) 1. Explain why this graph has an Eulerian chain.
There exists only 2 vertices with odd degree, vertices A and D.

2. What is the value of any Eulerian chain in this graph? ***28***

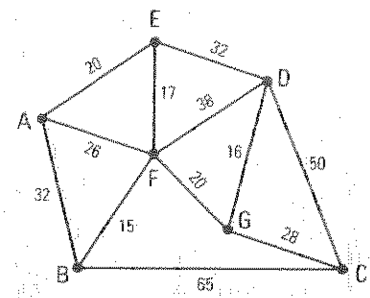
c) Find an Eulerian chain. ***ABEADECD***

d) Find the Hamiltonian chain with the smallest value.
CDABE or EBADC with value 11.

e) Find the Hamiltonian cycle starting at E. ***EBADCE or ECDABE***

9. The graph on the right illustrates a watering system in a garden. The vertices indicate the sprinklers and the edges represent pipes. The number written on the edges indicate the installation cost in dollars for the pipes. Which pipes should we plan in order to minimize the installation cost while keeping the system functional?

The pipes represented by edges AE, EF, FB, FG, GD and GC.

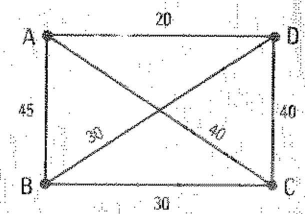


10. Julie leaves her office, represented by vertex A in the graph on the right, and must visit her clients represented by vertices B, C and D and then return to her office.

The numbers on the edges represent the distances (in km) between two vertices.

Knowing that Julie visits each client once and only once, determine Julie's shortest route.

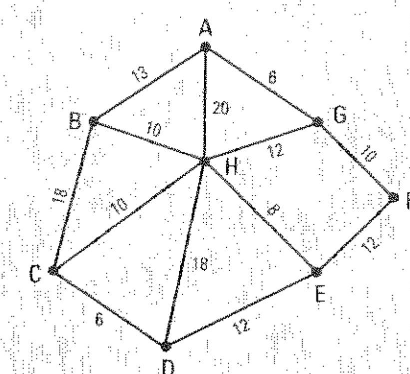
The Hamiltonian cycle ADBCA (120 km).



11. The vertices of the graph on the right represent the villages in a region and the edges represent roads connecting the villages.

The numbers on each edge represent the distances (in km) between each village. There is a fire in a house in village D. What route must the fire truck take in order to get to village D as quickly as possible if the fire station is located in village A?

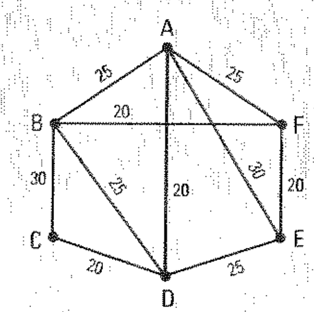
The route defined by the chain AGHCD (34 km).



12. Roger plans to visit a touristic region by car. He seeks the shortest itinerary passing through sites A, B, C, D, E and F. He starts at site A, wishes to visit site B before site C and return to site A at the end of his trip. The numbers on the edges in this graph indicate the distance (in km) between each site.

What is the itinerary that Roger must follow?

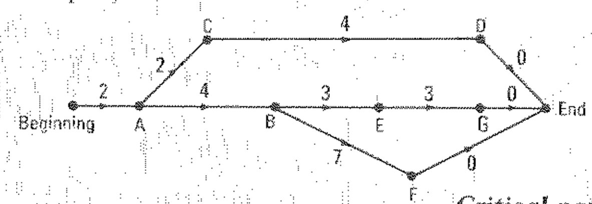
Itinerary AEFBCDA with length 140 km.



13. The realization of a project comprises several steps. The table on the right indicates the different steps, the completion time of each step as well as the prerequisite steps.

Step	Completion time	Prerequisite step(s)
A	2 days	None
B	4 days	A
C	2 days	A
D	4 days	C
E	3 days	B
F	7 days	B
G	3 days	E

a) What is, in days, the minimal time to complete this project?

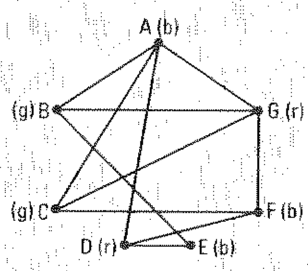


Critical path: Beginning ABF End. Duration: 13 days.

b) While carrying out the project, steps D and G took one extra day each. Did these additional days influence the minimal completion time for the project established in a)? ***No***

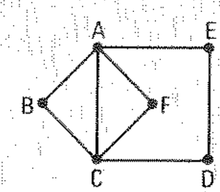
14. A meeting between seven countries is organized by the UN to resolve a conflict. The delegations from these countries must be accommodated in different hotels.
The countries are designated by letters. The table on the right indicates the delegations that do not want to stay in the same hotel.
Represent the situation using a graph and indicate the number of hotels necessary to satisfy the seven delegations.

Country	In conflict with country
A	B, C, D and G
B	A, G and E
C	A, F and G
D	A, F and E
E	B and D
F	C, G and D
G	A, B, C and F



Three hotels are necessary.

15. The graph on the right represents the trajectory of a light signal. The light signal has the property that it travels each edge of this graph in 5 seconds. The light signal starts at vertex A, moves each time towards vertex B, follows an Eulerian cycle and finally returns to vertex A and then leaves again.



a) What are the eight possible Eulerian cycles for the light signal?

b) How many Eulerian cycles does the signal travel in one hour?

90 cycles.

A	*B*	*C*	*F*	*A*	*E*	*D*	*C*	*A*
A	*B*	*C*	*F*	*A*	*C*	*D*	*E*	*A*
A	*B*	*C*	*A*	*F*	*C*	*D*	*E*	*A*
A	*B*	*C*	*A*	*E*	*D*	*C*	*F*	*A*
A	*B*	*C*	*D*	*E*	*A*	*C*	*F*	*A*
A	*B*	*C*	*D*	*E*	*A*	*F*	*C*	*A*

c) What is the probability that 30 seconds after it has started, the light signal is at vertex

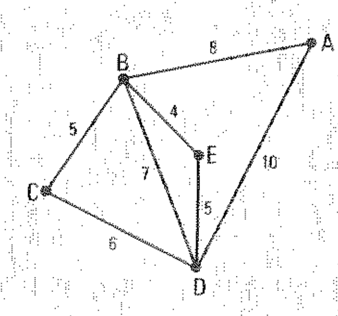
1. A *0* 2. C $\frac{1}{3}$ 3. D $\frac{1}{2}$ 4. F $\frac{1}{6}$

d) At a given instant of a cycle, the light signal is at a vertex. What is the probability that this vertex is

1. B? $\frac{1}{8}$ 2. C? $\frac{1}{4}$ 3. D? $\frac{1}{8}$

16. The graph on the right represents a neighbourhood in a city. The police station is located at vertex A. Each street of the neighbourhood is represented by an edge. A police car leaves the station at midnight and covers the neighbourhood by following an Eulerian cycle.

The time (in min) it takes for the car to travel along the street represented by this edge is written on each edge.



a) Find all directed Eulerian cycles that first visit vertex B.

ABCDEBDA; ABCDBEDA

ABDEBCDA; ABDCBEDA

ABEDBCDA; ABEDCBDA

b) How many directed Eulerian cycles that first visit vertex D are there in total? *6*

c) What is the probability that the police car reaches vertex E 20 minutes after midnight?

$\frac{1}{12}$

On a total of 12 directed Eulerian cycles, only one cycle allows the car to reach vertex E 20 minutes past midnight, cycle ABDEBCA.