Chapter 5

Probabilities

CHALLENGE 5

1. A card is drawn from a 52-card deck.

a) What is the probability of drawing a queen or a heart? $\frac{4}{13}$

b) If the card drawn is a heart, what is the probability that the card is a queen? $\frac{1}{13}$

c) If the card drawn is a queen, what is the probability that the card drawn is a heart?

$\frac{1}{4}$

2. According to the weather forecast, there is a 30% chance of having snow on Saturday, a 40% chance of having snow on Sunday and a 20% chance of having snow Saturday or Sunday. What is the probability that

a) there is snow on Sunday if it snows on Saturday? $\frac{2}{3}$

b) it snows neither on Saturday nor Sunday? 0.5

c) it doesn't snow on Sunday if it doesn't snow on Saturday? $\frac{5}{7}$

3. A coin is tossed 4 times. What is the probability of getting a total of 2 tails?

$\frac{3}{8}$

4. A die is rolled twice. Let X represent the total number of points obtained.

a) What is the most likely value for X? 7

b) What is the expected value of X? 7

c) What is the probability of getting a 6 on the 1st roll or on the 2nd roll? $\frac{11}{36}$

5. Hana goes to work either by car or by bus. She takes the car one in three times. She has a 20% chance of being late for work when she takes her car and a 15% chance of being late when she takes the bus.

Hana was late for work today. What is the probability that she took the car to go to work?

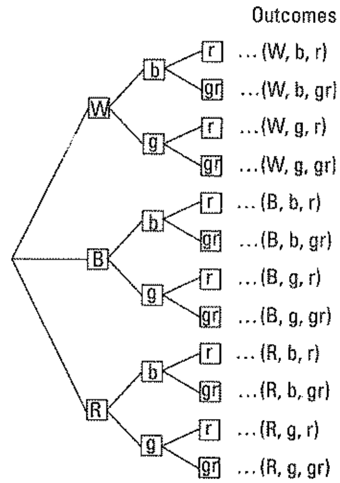
$\frac{2}{5}$

6. A game consists in rolling a die twice and receive, in dollars, twice the total number of points obtained. How much should a player pay to play this game so that it is fair?

$14

5.1 Arrangements and combinations

Activity 1 Tree



Eric owns 3 shirts: a white one (W), a black one (B) and a red one (R); 2 pairs of pants: black ones (b) and gray ones (g); 2 ties: a red one (r) and a green one (gr).

a) Complete the tree on the right showing the different ways Eric can dress.

b) How many ways does Eric have to choose

1. his shirt? ***3 ways***
2. his pants? ***2 ways***
3. his tie? ***2 ways***
4. his shirt and his tie? ***6 ways***
5. his shirt, pants and tie? ***12 ways***

c) To get dressed, Eric must go through 3 steps. The 1st step consists inchoosing a shirt, the 2nd step consists in choosing a pair of pants and the 3rd step consists in choosing his tie. Complete using the appropriate symbol.

Total number of ways to get dressed	=	Number of ways to perform the 1st step	×	Number of ways to perform the 2nd step	×	Number of ways to perform the 3rd step

FUNDAMENTAL PRINCIPLE

In any experiment requiring several steps (k steps),
- if the 1st step can be performed in n_1 ways,
- if the 2nd step can be performed in n_2 ways,

⋮
- if the kth step can be performed in n_k ways,

then the total number of ways to perform this experiment is equal to the product:

$$n_1 \times n_2 \times \ldots \times n_k$$

1. How many ways are there of composing a menu consisting of an appetizer, a main course and a dessert, if there are 3 choices for the appetizer, 4 choices for the main course and 2 choices for the dessert? ***24 ways***

2. Sixteen soccer teams compete in a tournament. How many ways are there to award gold, silver and bronze medals? ***3360 ways***

3. A license plate is composed of 3 letters followed by 3 digits. How many different license plates can there be? $26^3 \times 10^3 = 17\,576\,000$

4. How many possible outcomes can be observed in an experiment where

a) a coin is tossed twice? ***4***

b) a coin is tossed 3 times? ***8***

c) a die is rolled twice? ***36***

d) a coin is tossed n times? 2^n

ACTIVITY 2 Permutations

How many ways are there to seat

a) 3 people in a row of three chairs? ___6___ **b)** 4 people in a row of four chairs? ___24___

c) n people in a row of n chairs? ___$n(n-1)(n-2) \times \ldots \times 2 \times 1$___

PERMUTATION

- Consider 3 distinct objects a, b and c.
  Any ordered sequence of these three objects is called permutation.
  There is a total of 6 permutations: abc, acb, bac, bca, cab and cba.
- The total number of permutations of n objects is equal to the product:

$$n(n-1)(n-2) \times \ldots \times 2 \times 1$$

This product is designated by $n!$ (read: n factorial).
Ex.: The total number of permutations of 5 distinct objects a, b, c, d, and e is equal to 120.
Indeed, $5 \times 4 \times 3 \times 2 \times 1 = 120$.
We write: $5! = 120$.

5. How many ways are there to place 4 books on a shelf? ___24___

6. Four couples must be seated in a row of 8 chairs. How many ways can they sit

a) if each person wants to sit with his or her spouse? ___$4! \times 2! = 48$___

b) if there are no restrictions? ___$8! = 40\,320$___

7. Four boys and two girls must be seated in a row of 6 chairs. How many ways can they sit if

a) the boys want to sit together and the girls want to sit together?
___$2 \times 4! \times 2! = 96$___

b) the boys want to sit together? ___$3 \times 4! \times 2! = 144$___

c) the girls want to sit together? ___$5 \times 2! \times 4! = 240$___

d) there are no restrictions? ___$6! = 720$___

ACTIVITY 3 Arrangements and combinations

Two objects must be chosen among three distinct objects a, b and c. Describe the different choices possible when

a) order is taken into account and repetitions are allowed.
aa, ab, ac, ba, bb, bc, ca, cb, cc

b) order is taken into account and repetitions are not allowed.
ab, ac, ba, bc, ca, cb

c) order is not taken into account and repetitions are allowed.
aa, ab, ac, bb, bc, cc

d) order is not taken into account and repetitions are not allowed.
ab, ac, bc

ARRANGEMENTS – COMBINATIONS

When 2 objects are chosen among 3 distinct objects *a*, *b* and *c*, we distinguish the following four situations according to whether or not the order in which they are chosen is taken into account and whether or not repetitions are allowed.

- *Situation:* Order: yes Repetitions: yes
  Each of the 9 possible outcomes is called arrangement with repetition.

aa	*ab*	*ac*
ba	*bb*	*bc*
ca	*cb*	*cc*

- *Situation:* Order: yes Repetitions: no
  Each of the 6 possible outcomes is called arrangement without repetition.

•	*ab*	*ac*
ba	•	*bc*
ca	*cb*	•

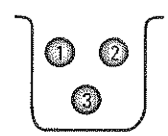
- *Situation:* Order: no Repetitions: yes
  Each of the 6 possible outcomes is called combination with repetition.

aa	*ab*	*ac*
•	*bb*	*bc*
•	•	*cc*

- *Situation:* Order: no Repetitions: no
  Each of the 3 possible outcomes is called combination without repetition.

•	*ab*	*ac*
•	•	*bc*
•	•	•

8. A jar contains 3 balls numbered 1, 2 and 3.
In each of the following situations, indicate if the possible outcomes are arrangements with or without repetition or combinations with or without repetition.

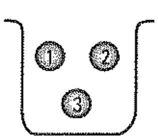


a) Two balls are drawn from the jar **successively with replacement** and the number on the ball is noted after each draw.
The possible outcomes are arrangements with repetition.

b) Two balls are drawn from the jar **successively without replacement** and the number on the ball is noted after each draw.
The possible outcomes are arrangements without repetition.

c) Two balls are drawn from the jar **simultaneously** and the number on the two balls is noted.
The possible outcomes are combinations without repetition.

9. Consider the jar in exercise 8. Make a list of possible outcomes if



a) two balls are drawn **successively with replacement** from the jar.
(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)

b) two balls are drawn **successively without replacement**. ***(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)***

c) two balls are drawn **simultaneously**. ***{1,2}, {1,3}, {2,3}***

10. David must choose 2 friends among 5 friends to go to the movies.

a) Is the order in which he chooses the 2 friends important? ***No***

b) Can he choose the same friend twice? ***No***

c) How many choices does he have? ***10***

11. From the board of directors of a company, comprised of 10 people, 2 people must be chosen to become chairman and vice-president of the company.

a) When enumerating the possible choices in this situation,

1. must order be taken into account? ***Yes*** 2. are repetitions allowed? ***No***

b) How many possible choices are there? ***90***

12. In a company's union, the positions of treasurer and secretary are vacant. A person is allowed to occupy both these positions. Four candidates apply.

a) When enumerating the possible choices in this situation,

1. must order be taken into account? ***Yes*** 2. are repetitions allowed? ***Yes***

b) How many possible choices are there? ***16***

13. A jury composed of 3 men and 2 women, chosen among 4 men and 5 women, must be formed.

a) How many ways are there of choosing the men who are member of the jury? ***4***

b) How many ways are there of choosing the women who are member of the jury? ***10***

c) How many possible juries are there? ***40***

14. Two students must be chosen among 3 boys and 3 girls.

a) How many possible choices are there? ***15***

b) How many possible choices are there if we must choose

1. one girl? ***9*** 2. no girls? ***3***
3. 2 girls? ***3*** 4. at least one girl? ***12***

15. a) The total number of **arrangements without repetition** when k objects are chosen among n distinct objects is equal to: $\frac{n!}{(n-k)!}$.

1. How many ways are there to choose 4 objects among 10 distinct objects when order is taken into account? ***5040 ways***

2. How many ways are there to award gold, silver and bronze medals at a tennis tournament where 8 players remain in the quarter-finals? ***336 ways***

b) The total number of **combinations without repetition** when k objects are chosen among n distinct objects is equal to: $\frac{n!}{k!(n-k)!}$.

1. How many ways are there to choose 4 objects among 10 distinct objects when order is not taken into account? ***210 ways***

2. How many ways are there to play 6/49 where 6 numbers must be chosen among 49 numbers? ***13 983 816 ways***

5.2 Voting procedures

Activity 1 Voting procedures

The 40 members of the board of directors of a company must vote to elect the president of the board. Three candidates Ann, Bernard and Caroline apply for the position. Each voting member must write, in order, his preference for the choice of each candidate.

Votes	16	14	10
1st choice	A	C	B
2nd choice	B	A	A
3rd choice	C	B	C

The results are compiled in the table on the right. Thus, we observe that 16 members chose the order (A, B, C), 14 chose the order (C, A, B), 10 chose the order (B, A, C) and none chose the order (A, C, B) or (C, B, A) or (B, C, A).

a) In a **majority ballot**, the winner is the one who gets more than half the votes.

1. Is there a candidate who wins in a majority ballot? ***No***
2. On how many ballots, at the minimum, must a candidate be rated as 1st choice to win a majority ballot? ***21***

b) In a **plurality ballot**, the winner is the one who gets the most votes, in other words the one who has been rated as 1st choice most often.

Which of the three candidates has been rated as 1st choice most often and would win a plurality ballot? ***Ann***

c) **Borda's method** is a procedure in which points are allocated to each candidate. If, for example, 2 points are allocated to the preferred candidate, 1 point to the following one and 0 to the last one, the winner is the candidate who earns the most points.

Determine for each candidate the total number of points and then deduce the winner according to Borda's method.

Ann earns 56 points (16 × 2pts + 14 × 1pt + 10 × 1pt), Bernard earns 36 points (16 × 1pt + 14 × 0pt + 10 × 2pts), Caroline earns 28 points (16 × 0pt + 14 × 2pts +10 × 0pt). Ann wins.

d) Under **Condorcet's criterion**, the winner is the one who wins over the other candidates in a head-to-head confrontation.

1. Determine which candidate would be the winner in a confrontation between
   1) Ann and Bernard. ***Ann is the winner with 30 votes.***
   2) Ann and Caroline. ***Ann is the winner with 26 votes.***
   3) Bernard and Caroline ***Bernard is the winner with 26 votes.***
2. Which candidate would therefore be the winner under Condorcet's criterion? ***Ann***

e) In an **elimination ballot**, the winner is determined by the following procedure:

1. In the 1st step, first place votes are counted for each candidate and the one who holds the least number of votes is eliminated.

   Determine the 1st place votes for each candidate and verify that Bernard is eliminated.

   Ann gets 16 votes, Bernard 10 votes and Caroline 14 votes. Therefore, Bernard is eliminated.

2. The 2nd step consists in eliminating, from the preference table, the candidate that was eliminated in the 1st step and to allocate the first place votes of the eliminated candidate to the candidate that follows him, and then to recount the first place votes.

   Bernard being eliminated, Ann, who follows him in the choice (B, A,C) will reap Bernard's 10 first place votes. Ann therefore gets a total of 26 first place votes (16 + 10),which represents a majority of votes. Who is therefore the winner under an elimination vote? ***Ann***

VOTING PROCEDURES

Let us illustrate the different voting procedures with the following example:

Three villages A, B and C in a region are candidates to determine the village where a school will be built. The results of a poll are represented in the preference table on the right.

45 %	35 %	20 %
A	B	C
B	A	B
C	C	A

There exists different voting procedures:

- **Majority ballot:**
  The winning candidate is the one who received more than half the votes.
  Ex.: None of the villages obtains a majority. We do not get a winning village under a majority ballot.
- **Plurality ballot:**
  The winning candidate is the one who received the greatest number of votes.
  Ex.: Village A who received the greatest number of votes is the winner under a plurality ballot.
- **Borda's method:**
  Weighted voting system. Each voter constructs a list of n candidates in order of preference. For example, if there are 4 candidates, we allocate 3 points to the first one, 2 points to the second one, etc... and 0 points to the last one. The candidate(s), whose score is the greatest, wins the election.
  Ex.: Let us allocate 2 points to the village in the 1st place, 1 point to the 2nd place and 0 point to the 3rd place. A receives 125 points ($45 \times 2 + 35 \times 1 + 20 \times 0$), B receives 135 points ($45 \times 1 + 35 \times 2 + 20 \times 1$), C receives 40 points ($45 \times 0 + 35 \times 0 + 20 \times 2$). Village B is the winner under Borda's method.
- **Condorcet's criterion:**
  Voting system in which the sole winner is the one, if he exists, who, when compared to each other candidate in turn, proves to be the preferred candidate every time.
  Ex.: If villages B and A are confronted, B wins with 55%. If B and C are confronted, B wins with 80%. Village B is the preferred candidate every time and is declared the winner under Condorcet's criterion.
- **Elimination ballot:**
  In the 1st step,
  - we count for each candidate the number of 1st place votes,
  - we reject the one who holds the fewest votes.

In the 2nd step,
- we eliminate the rejected candidate from the preference table,
- we allocate the number of 1st place votes of the rejected candidate to the following candidate,
- we recount the number of 1st place votes.

If a candidate gets a majority of votes, he wins the election. Otherwise, we eliminate the one who has the fewest votes and we repeat the procedure until we obtain a winner.

Ex.: We reject candidate C since it obtained only 20% of the 1st place votes. Candidate B which follows then receives an additional 20% of the 1st place votes and thus obtains 55% of the 1st place votes in the new preference table. Village B is declared the winner under an elimination vote.

45 %	55 %
A	B
B	A

1. The 20 members of the board of directors of a company must vote to elect the president of the board. Three candidates Andy, Bridget and Clara apply for the position. Each voting board member must write, in order of preference, his choice of candidate.

a) How many distinct results in total can be observed in this vote?

6 × 3! distinct results

The 20 results have been compiled in the following preference table:

Votes	4	3	2	3	5	3
1st choice	A	A	B	B	C	C
2nd choice	B	C	A	C	A	B
3rd choice	C	B	C	A	B	A

b) Determine the winner, and justify your answer, under

1. a majority ballot. ***No winner. A(7), B(5), C(8).***
   At least 21 votes are necessary to get elected.
2. a plurality ballot. ***Clara with eight 1st place votes.***
3. Borda's method. ***A (21 pts), B (17 pts), C (22 pts). Clara is elected.***
4. Condorcet's criterion. ***C wins 11 times against A and C wins 11 times against B.***
   Clara is elected.
5. an elimination ballot. ***B is eliminated. C then receives eleven 1st place votes.***
   Clara is elected.

2. Three candidates *a*, *b* and *c*, apply for the position of president of a company. The 10 members of the board of directors have ranked the candidates in order of their preference.

The table on the right gives the results. Is the elected candidate the same whether a majority ballot or Borda's method is used?

Votes	6	3	1
1st choice	*a*	*b*	*c*
2nd choice	*b*	*c*	*b*
3rd choice	*c*	*a*	*a*

No, under a majority ballot, candidate a is elected while under Borda's method, it is candidate b who is elected.

3. Three candidates a, b and c apply for the position of secretary of a union. The results are given in the preference table on the right.

Is the elected candidate the same whether Condorcet's criterion or a plurality ballot is used?

Votes	33 %	29 %	26 %	12 %
1st choice	a	b	c	c
2nd choice	c	c	b	a
3rd choice	b	a	a	b

No, under Condorcet's criterion, candidate c is elected while under a plurality ballot, it is candidate a who is elected.

4. Three candidates a, b and c are running to become class president. The results are given in the preference table on the right.

Is the elected candidate the same whether Condorcet's criterion or Borda's method is used?

Votes	16	8	4
1st choice	a	b	c
2nd choice	b	c	b
3rd choice	c	a	a

No, under Condorcet's criterion, candidate a is elected while under Borda's method, it is candidate b who is elected.

5. The 40 members of the board of directors of a company must vote to elect the president of the board. Four candidates Ann, Bob, Cindy and David apply for the position. Each voting member must write his order of preference for each candidate.

a) How many distinct results in total can be observed in this vote?

24 (4!) distinct results.

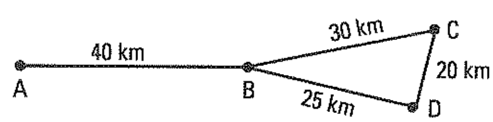
The 40 results have been compiled in the following preference table:

Votes	18	16	4	2
1st choice	A	B	C	D
2nd choice	B	C	B	B
3rd choice	C	D	D	C
4th choice	D	A	A	A

b) Determine the winner under

1. a majority ballot. ***0***
2. a plurality ballot. ***A***
3. Borda's method. ***B***
4. Condorcet's criterion. ***B***
5. an elimination ballot. ***B***

6. We want to determine in which one of the four villages on the right a medical centre will be built. Each resident wishes that the centre be closest to his village. A survey gave the preference table on the right.



Determine the winning village under

a) a majority ballot. ***None***

b) a plurality ballot. ***A***

c) Borda's method. ***B***

d) Condorcet's criterion. ***B***

e) an elimination ballot. ***B***

Village A (40 %)	Village B (35 %)	Village C (15 %)	Village D (10 %)
A	B	C	D
B	D	D	C
D	C	B	B
C	A	A	A

7. For each of the following preference tables, determine the winner of the vote according to the voting procedure.

a)

Votes	6	7	7
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *0*
2. a plurality ballot. *B or C*
3. Borda's method. *C*
4. Condorcet's criterion. *0*
5. an elimination ballot. *B*

b)

Votes	9	3	8
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *A*
4. Condorcet's criterion. *0*
5. an elimination ballot. *C*

c)

Votes	7	11	2
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *B*
2. a plurality ballot. *B*
3. Borda's method. *B*
4. Condorcet's criterion. *B*
5. an elimination ballot. *B*

d)

Votes	9	5	6
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *A*
4. Condorcet's criterion. *0*
5. an elimination ballot. *C*

e)

Votes	9	7	4
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *B*
4. Condorcet's criterion. *0*
5. an elimination ballot. *A*

f)

Votes	7	7	6
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

1. a majority ballot. *0*
2. a plurality ballot. *0*
3. Borda's method. *0*
4. Condorcet's criterion. *0*
5. an elimination ballot. *A*

g)

Votes	8	7	5
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *B*
4. Condorcet's criterion. *B*
5. an elimination ballot. *C*

h)

Votes	9	8	3
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *B*
4. Condorcet's criterion. *B*
5. an elimination ballot. *C*

i)

Votes	9	6	5
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

1. a majority ballot. *0*
2. a plurality ballot. *A*
3. Borda's method. *B*
4. Condorcet's criterion. *B*
5. an elimination ballot. *B*

5.3 Events

ACTIVITY 1 Universal set – Events

A fair die is rolled once.

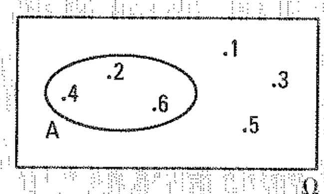
a) Describe the universal set Ω, in other words the set of all possible outcomes.
Ω = {1, 2, 3, 4, 5, 6}

b) Describe the event
1. A: «rolling an even number». *A = {2, 4, 6}*
2. B: «rolling a number greater than or equal to 1» *B = {1, 2, 3, 4, 5, 6}*
3. C: «rolling a number greater than 6» *C = ∅*

c) Which of the three events defined in b) is
1. certain. *B* 2. impossible. *C*

UNIVERSAL SET – EVENT

- An experiment is random when the outcome of this experiment cannot be predicted with certainty.
  Ex.: The experiment consisting in rolling a die and observing the outcome is a random experiment.
- The universal set associated with a random experiment is the set of all possible outcomes of the experiment.
  This set is written Ω (read «omega»).
  Ex.: The possible outcomes of the experiment above are 1, 2, 3, 4, 5 or 6.
  Therefore, we have Ω = {1, 2, 3, 4, 5, 6}.
- An event associated with a random experiment is a subset of the set Ω of all possible outcomes.
  Ex.: A die is rolled once.
  The event A: «rolling an even number» is described extensively by A = {2, 4, 6} and on the right using a Venn diagram.



- Among events, we distinguish
  – the certain event, written Ω, event which always occurs.
  – the impossible event, written ∅, event which never occurs.

1. For each of the following random experiments, determine the universal set Ω. (Use a tree diagram if necessary).

a) A coin is tossed twice.
Ω = {(T, T), (T, H), (H, T), (H, H)}

b) A die is rolled twice.
Ω = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), ..., (2,6), (3,1), ..., (3,6), (4,1), ..., (4,6), (5,1), ..., (5,6), (6,1), ..., (6,6)}

c) A coin is tossed three times.
Ω = {(T, T, T), (T, T, H), (T, H, T), (T, H, H), (H, T, T), (H, T, H), (H, H, T), (H, H, H)}

2. A die is rolled twice.

a) Describe extensively the following events.

1. A: «rolling a sum equal to 10». ***A = {(4,6), (5,5), (6,4)}***
2. B: «rolling a product equal to 12». ***B = {(2,6), (3,4), (4,3), (6,2)}***
3. C: «getting an even number on the first roll and a sum equal to 7». ***C = {(2,5), (4,3), (6,1)}***

b) Describe in words the following events.

1. A = {(6,1), (6,2), (6,3) (6,4), (6,5), (6,6)} ***«Getting a 6 on the 1st roll.»***
2. B = {(1,4), (2,3), (3,2) (4,1)} ***«Rolling a sum equal to 5.»***
3. C = {(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)} ***«Rolling a sum less than 5.»***

3. Two people chosen at random on a street corner are asked about their opinion regarding the merger of two cities. Each person can declare that they agree (a), they disagree (d) or refuse to answer (r).

a) Describe extensively the universal set Ω.

Ω = {(a, a), (a, d), (a, r), (d, a), (d, d), (d, r), (r, a), (r, d), (r, r)}

b) Describe extensively the following events.

A: «the two people agree». ***A = {(a, a)}***

B: «the 1st person agrees». ***B = {(a, a), (a, d), (a, r)}***

C: «the two people give the same answer». ***C = {(a, a), (d, d), (r, r)}***

D: «at least one of the two people agrees». ***D = {(a, a), (a, d), (a, r), (d, a), (r, a)}***

c) Describe in words the following events.

A = {(r, r)} ***«The two people refuse to answer.»***

B = {(a, d), (d, d), (r, d)} ***«The 2nd person disagrees.»***

C = {(a, d)} ***«The 1st person agrees and the second disagrees.»***

D = {(d, d), (d, r), (r, d), (r, r)} ***«Neither person agrees.»***

4. At basketball, Valerie shoots the ball at the basket three times consecutively. Let S represent a success and F a failure.

a) Describe extensively the universal set Ω of all possible outcomes.

Ω = {(S, S, S), (S, S, F), (S, F, S), (S, F, F), (F, S, S), (F, S, F), (F, F, S), (F, F, F)}

b) Describe extensively the following events:

A: «Valerie succeeds on her 1st shot and fails on the other two.». ***A = {(S, F, F)}***

B: «Valerie succeeds on each shot.». ***B = {(S, S, S)}***

C: «Valerie succeeds on the first shot.». ***C = {(S, S, S), (S, S, F), (S, F, S), (S, F, F)}***

c) Describe in words the following events.

A = {(F, F, S)} ***«Valerie succeeds only on her 3rd shot.»***

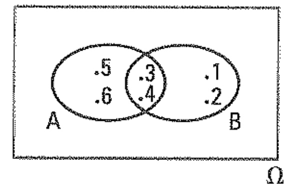
B = {(F, F, F)} ***«Valerie fails on each shot.»***

C = {(S, S, F), (S, F, F), (F, S, F), (F, F, F)} ***«Valerie fails on her 3rd shot.»***

5.4 Operations between events

ACTIVITY 1 The event A and B

A die is rolled once. Consider the following events.
A: "rolling a number greater than 2".
B: "rolling a number less than 5".



a) Describe extensively the following events.

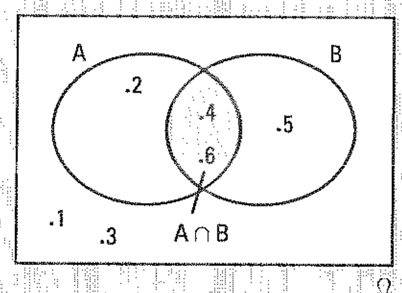
1. A = *{3, 4, 5, 6}* 2. B = *{1, 2, 3, 4}*

b) What is the set of possible outcomes if we know that both events A and B occurred? *{3, 4}*

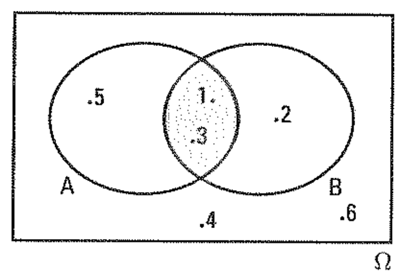
INTERSECTION

Given a random experiment and two events A and B, the event A and B, written $A \cap B$ (read: A intersect B), is the event that occurs if, and only if, A occurs and B occurs.

Ex.: A die is rolled once. Consider the following events.
A: "rolling an even number".
B: "rolling a number greater than 3".
The event $A \cap B$ is defined by: "rolling an even number greater than 3".
Note that:
A = {2, 4, 6}, B = {4, 5, 6} and $A \cap B$ = {4, 6}.



1. A die is rolled once. Let A and B be the events defined by:
A: "rolling an odd number"; B: "rolling a number less than 4".



a) Represent the events Ω, A and B on a Venn diagram.

b) 1. Color the region associated with the event $A \cap B$ on this diagram.
2. Describe extensively the event $A \cap B$. *{1, 3}*
3. Describe in words the event $A \cap B$. ***"Rolling an odd number less than 4."***

2. A coin is tossed twice.

a) Describe extensively the following events.
1. A: "getting tails on the 1st toss". *{(T, T), (T, H)}*
2. B: "getting tails on the 2nd toss". *{(H, T), (T, T)}*

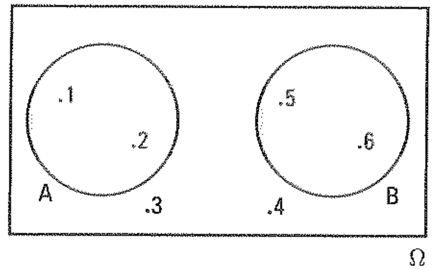
b) Describe in words the event $A \cap B$. ***"Getting tails on each toss"***

c) Describe extensively the event $A \cap B$. *{(T, T)}*

3. A card is drawn from a 52-card deck. Let A and B be two events defined by:
A: "drawing a queen"; B: "drawing a heart".
Describe in words the event $A \cap B$. ***"Drawing the queen of hearts."***

ACTIVITY 2 Incompatible events

A die is rolled once. Let A and B be the events defined by:
A: "rolling a number less than 3"; B: "rolling a number greater than 4".



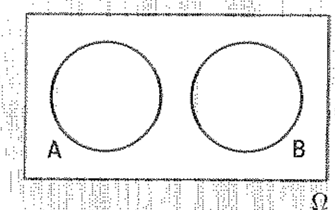
a) Describe extensively the following events.
1. A = ***{1, 2}*** 2. B = ***{5, 6}***

b) Represent the events Ω, A and B on a Venn diagram.

c) 1. Describe in words the event $A \cap B$. ***"Rolling a number less than 3 and greater than 4."***
2. What can you say about the event $A \cap B$? ***$A \cap B$ is an impossible event, since $A \cap B = \varnothing$.***

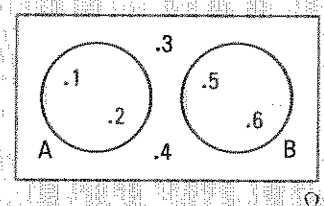
INCOMPATIBLE EVENTS

Two events A and B are incompatible (or mutually exclusive) if it is impossible for these two events to occur simultaneously. $A \cap B$ is then the impossible event.



Thus, A and B are incompatible $\Leftrightarrow A \cap B = \varnothing$

Ex.: In the experiment where a die is rolled once, the events
A: "rolling a number less than or equal to 2"
and B: "rolling a number greater than or equal to 5" are incompatible since $A \cap B = \varnothing$.



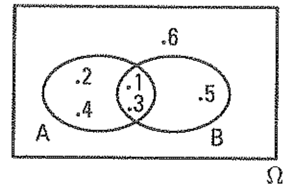
4. A fair die is rolled once. Let A, B, C and D be the events defined by;
A: "rolling an even sum"; B: "rolling a sum equal to 5";
C: "rolling a sum greater than 10"; D: "getting 6 on the 1st roll".
True or false?

a) A and B are incompatible. ***T***
b) A and C are incompatible. ***F***
c) A and D are incompatible. ***F***
d) B and C are incompatible. ***T***
e) B and D are incompatible. ***T***
f) C and D are incompatible. ***F***

ACTIVITY 3 Event A or B

A die is rolled once. Consider the following events.

A: "rolling a number less than 5" and B: "rolling an odd number".



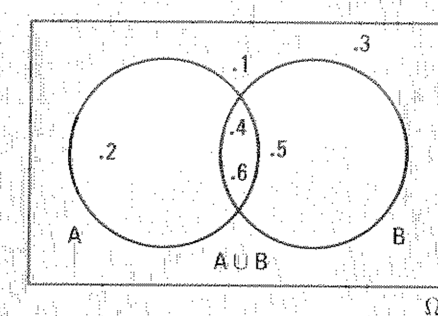
a) Describe extensively the following events.

1. A = ***{1, 2, 3, 4}***
2. B = ***{1, 3, 5}***

b) What is the set of all possible outcomes if we know that event A occurred or that event B occurred? ***{1, 2, 3, 4, 5}***

UNION

Given a random experiment and two events A and B, the event A or B, written $A \cup B$ (read: A union B), is the event that occurs if, and only if, A occurs or B occurs.



Ex.: A die is rolled once. Consider the following events.
A: "rolling an even number".
B: "rolling a number greater than 3".
The event $A \cup B$ is defined by: "rolling an even number or a number greater than 3".
Note that: A = {2, 4, 6}, B = {4, 5, 6} and $A \cup B$ = {2, 4, 5, 6}.

5. A die is rolled twice. Let A and B be the events defined by:
A: "getting 6 on the 1st roll"; B: "getting 6 on the 2nd roll".

Describe extensively the following events.

a) A ***{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}***

b) B ***{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)}***

c) $A \cup B$ ***{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6)}***

d) $A \cap B$ ***{(6, 6)}***

6. Nam Soo writes an English exam and a French exam. We observe, for each exam, his performance, namely if he passes of fails. The couple (F, S) indicates that he failed in English and passed in French.

a) Describe extensively the following events.

1. A: "Nam Soo passes the English exam". ***{(S, S), (S, F)}***
2. B: "Nam Soo passes the French exam". ***{(F, S), (S, S)}***

b) Describe in words the following events.

1. $A \cap B$ ***"Passing both exams."***
2. $A \cup B$ ***"Passing the English exam or the French exam."***
   Other answer: "Passing at least one of the two exams."

c) Describe extensively the following events.

1. $A \cap B$ ***{(S, S)}***
2. $A \cup B$ ***{(S, F), (F, S), (S, S)}***

ACTIVITY 4 Event contrary to event A

The contrary event to an event A is designated by $\bar{A}$.
A die is rolled once.

a) Describe in words the contrary to the following events.

1. A_1: "rolling an even number". ***"Rolling an odd number."***
2. A_2: "rolling a 3". ***"Rolling a number different from 3."***
3. A_3: "rolling a number less than 3". ***"Rolling a number greater than or equal to 3."***

b) Describe extensively the contrary to the following events.

1. $A_1 = \{2, 4, 6\}$ ***$\bar{A}_1 = \{1, 3, 5\}$*** 2. $A_2 = \{3\}$ ***$\bar{A}_2 = \{1, 2, 4, 5, 6\}$*** 3. $A_3 = \{1, 2\}$ ***$\bar{A}_3 = \{3, 4, 5, 6\}$***

CONTRARY EVENT

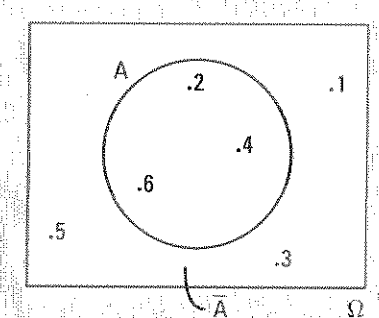
Consider an event A associated with a random experiment. The event $\bar{A}$ (read: A bar) is the contrary event (complementary) to event A.

$\bar{A}$ occurs if, and only if, A does not occur.

The event $\bar{A}$ is sometimes written A′ (read: A prime).

Ex.: A die is rolled once. Consider the event A: "rolling an even number".
The event $\bar{A}$ is defined by: "rolling an odd number".
Note that $A = \{2, 4, 6\}$ and $\bar{A} = \{1, 3, 5\}$.

We have: $A \cap \bar{A} = \emptyset$ and $A \cup \bar{A} = \Omega$.



7. A die is rolled once. Let A_1, A_2, A_3 be the events defined by:
A_1: "rolling an odd number"; A_2: "rolling a 5"; A_3: "rolling a number greater than 3".

a) Describe in words the following events.

1. $\bar{A}_1$ ***"Rolling an even number."***
2. $\bar{A}_2$ ***"Rolling a number different from 5."***
3. $\bar{A}_3$ ***"Rolling a number less than or equal to 3."***

b) Describe extensively the following events.

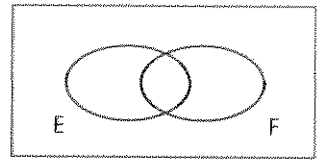
1. $\bar{A}_1$ ***{2, 4, 6}*** 2. $\bar{A}_2$ ***{1, 2, 3, 4, 6}*** 3. $\bar{A}_3$ ***{1, 2, 3}***

ACTIVITY 5 De Morgan's laws

Consider, in a group of tourists, the set E of tourists who speak English and the set F of those who speak French.

a) 1. Represent $\overline{E \cup F}$ on the diagram on the right.

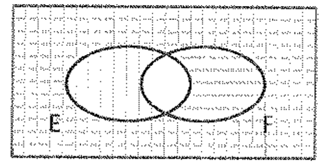
2. Describe in words $\overline{E \cup F}$. ***Set of tourists who speak neither English nor French.***



b) On the diagram on the right,

1. shade $\overline{E}$ using horizontal lines. 2. shade $\overline{F}$ using vertical lines.

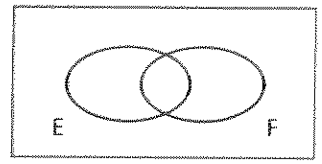
3. how is the set $\overline{E} \cap \overline{F}$ represented? ***By a grid***



c) Compare $\overline{E \cup F}$ and $\overline{E} \cap \overline{F}$. ***$\overline{E \cup F} = \overline{E} \cap \overline{F}$***

d) 1. Represent $\overline{E \cap F}$ on the diagram on the right.

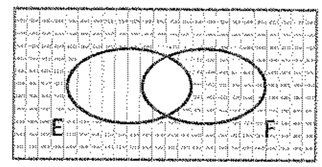
2. Describe $\overline{E \cap F}$ in words. ***Set of tourists who are not bilingual.***



e) On the diagram on the right,

1. shade $\overline{E}$ using horizontal lines. 2. shade $\overline{F}$ using vertical lines.

f) Compare $\overline{E \cap F}$ and $\overline{E} \cup \overline{F}$. ***$\overline{E \cap F} = \overline{E} \cup \overline{F}$***



DE MORGAN'S LAWS

Let A and B be two events associated with a random experiment. We have:

$\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Thus, the complement of the union is equal to the intersection of the complements and the complement of the intersection is equal to the union of the complements.

8. A die is rolled once. Consider the following events.
A: "rolling an even number" and B: "rolling a number greater than 3".

a) Describe extensively the following events.

1. $A \cup B$ ***{2, 4, 5, 6}*** 2. $A \cap B$ ***{4, 6}*** 3. $\overline{A \cup B}$ ***{1, 3}*** 4. $\overline{A \cap B}$ ***{1, 2, 3, 5}***

b) Describe extensively the following events.

1. $\overline{A}$ ***{1, 3, 5}*** 2. $\overline{B}$ ***{1, 2, 3}*** 3. $\overline{A} \cup \overline{B}$ ***{1, 2, 3, 5}*** 4. $\overline{A} \cap \overline{B}$ ***{1, 3}***

c) Verify De Morgan's two laws.

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ 2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

d) Describe in words the following events.

1. $\overline{A} \cap \overline{B}$ ***"Rolling an odd number less than or equal to 3."***
2. $\overline{A \cap B}$ ***"Not rolling an even number greater than 3."***
3. $\overline{A} \cup \overline{B}$ ***"Rolling a number that is odd or less than or equal to 3."***
4. $\overline{A \cup B}$ ***"Not rolling a number that is even or greater than 3."***

9. A student is chosen at random from a group. Consider the following events.
F: "the student is a girl" and G: "the student wears glasses".

Describe in words the following events.

a) $F \cup G$ ***"The student is a girl or wears glasses."***

b) $F \cap G$ ***"The student is a girl and wears glasses."***

c) $\overline{F}$ ***"The student is a boy"***

d) $F \cap \overline{G}$ ***"The student is a girl and does not wear glasses."***

e) $\overline{F} \cap \overline{G}$ ***"The student is a boy and does not wear glasses."***

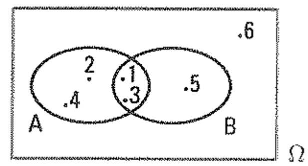
f) $\overline{F} \cup \overline{G}$ ***"The student is a boy or does not wear glasses."***

ACTIVITY 6 The event A minus B

A die is rolled once. Consider the following events.
A: "rolling a number less than 5" and B: "rolling an odd number".

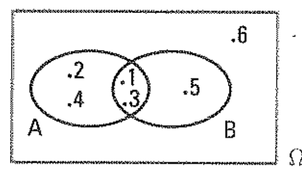
a) 1. Describe extensively the event $A \cap \overline{B}$. ***{2, 4}***

2. Describe in words the event $A \cap \overline{B}$.
"Rolling an even number less than 5".



b) 1. Describe extensively the event $\overline{A} \cap B$. ***{5}***

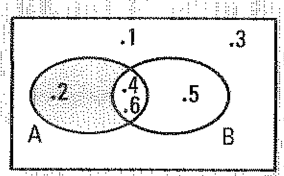
2. Describe in words the event $\overline{A} \cap B$.
"Rolling an odd number greater than or equal to 5."



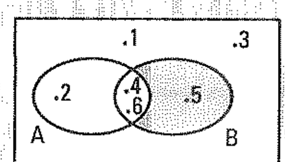
DIFFERENCE EVENT

Given a random experiment and two events A and B, the event "A minus B", written $A \setminus B$ (read: A minus B), is the event that occurs if, and only if, A occurs and B does not occur.

Ex.: A die is rolled once. Consider the following events.
A: "rolling an even number" and B: "rolling a number greater than 3".
– The event $A \setminus B$ is defined by: "rolling an even number less than or equal to 3".
– The event $B \setminus A$ is defined by: "rolling an odd number greater than 3".



$A \setminus B = \{2\}$



$B \setminus A = \{5\}$

We have: $A \setminus B = A \cap \overline{B}$ and $B \setminus A = \overline{A} \cap B$

10. A student is chosen at random from a group writing English and biology exams. Consider the following events.

A: "the student passes in English" and B: "the student passes in biology".

Describe in words the following events.

a) $A \cup B$ ***"The student passes at least one of the two exams."***

b) $A \cap B$ ***"The student passes both exams."***

c) $\overline{A}$ ***"The student fails the English exam."***

d) $A \setminus B$ ***"The student passes in English only."***

e) $B \setminus A$ ***"The student passes in biology only."***

f) $\overline{A} \cap \overline{B}$ ***"The student fails both exams."***

g) $\overline{A} \cup \overline{B}$ ***"The student fails at least one of the two exams."***

11. A student from the class is chosen at random. Consider the following events.

B: "the student is a boy" and R: "the student is right-handed".

Write the following events symbolically.

a) The student is a right-handed boy. $B \cap R$

b) The student is a left-handed boy. $B \cap \overline{R}$ ***or*** $B \setminus R$

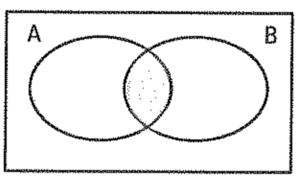
c) The student is a right-handed girl. $\overline{B} \cap R$ ***or*** $R \setminus B$

d) The student is a left-handed girl. $\overline{B} \cap \overline{R}$ ***or*** $\overline{B \cup R}$

12. Let A and B be two events associated with a random experiment.

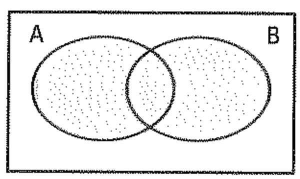
1. Represent the following events on a Venn diagram.
2. Describe each event using the operators $\cap$, $\cup$ or $\overline{}$.

a) A and B occur.



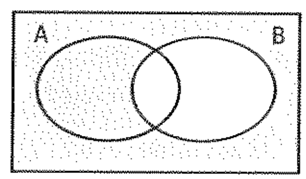
$A \cap B$

b) A or B occurs.



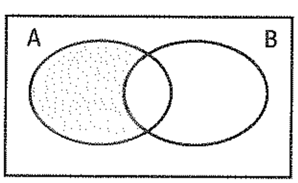
$A \cup B$

c) B does not occur.



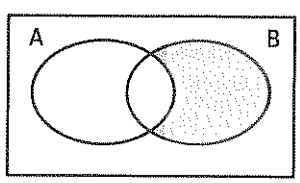
$\overline{B}$

d) A occurs and B does not occur.



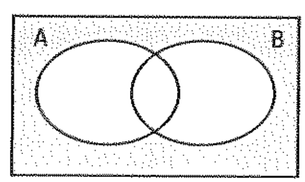
$A \cap \overline{B}$

e) A does not occur and B occurs.



$\overline{A} \cap B$

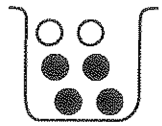
f) A and B do not occur.



$\overline{A \cup B}$ ***or*** $\overline{A} \cap \overline{B}$

5.5 Probability of an event

ACTIVITY 1 Probability of an outcome



A jar contains 6 balls: 4 black ones and 2 white ones. A ball is drawn from the jar and its color is recorded. W represents the outcome "white" and B the outcome "black".

a) Describe the set Ω of all possible outcomes. *$\Omega = \{W, B\}$*

b) Are the outcomes equally likely? *No*

c) Calculate

1. the probability of drawing a black ball, written P(B). *$P(B) = \frac{2}{3}$*
2. the probability of drawing a white ball, written P(W). *$P(W) = \frac{1}{3}$*

d) If the jar contains 6 balls: 3 black ones and 3 white ones, answer questions a), b) and c) again.

a) *$\Omega = \{W, B\}$* **b)** *Yes* **c)** *$P(W) = P(B) = \frac{1}{2}$*

ACTIVITY 2 Probability of an event

A die is rolled once and the outcome is recorded.

a) Describe the set Ω of possible outcomes. *$\Omega = \{1, 2, 3, 4, 5, 6\}$*

b) Are the possible outcomes equally likely? *Yes*

c) Consider the event A: "rolling an even number".

1. Describe event A extensively. *$A = \{2, 4, 6\}$*
2. Calculate the probability, written P(A), that event A occurs. *$P(A) = \frac{1}{2}$*

d) Consider the event B: "rolling a number less than 5".

1. Describe event B extensively. *$B = \{1, 2, 3, 4\}$*
2. Calculate P(B). *$P(B) = \frac{2}{3}$*

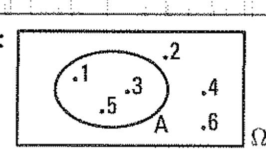
PROBABILITY OF AN EVENT IN AN EQUALLY LIKELY SITUATION

Let Ω be the universal set associated with a random experiment and A an event. If the possible outcomes of the experiment are equally likely, the probability of event A, written P(A), is the ratio between the number of outcomes favourable to the occurrence of A and the number of possible outcomes.

$$P(A) = \frac{\text{Number of outcomes favourable to event A}}{\text{Number of possible outcomes}} = \frac{n(A)}{n(\Omega)}$$

Note: n(E) represents the number of elements in E, called cardinality of E.

Ex.:



A die is rolled once.

A: "rolling an odd number".

$A = \{1, 3, 5\}$, $P(A) = \frac{3}{6} = \frac{1}{2}$

1. A card is drawn from a 52-card deck. What is the probability of drawing

a) a king? $\frac{1}{4}$ b) a spade? $\frac{1}{13}$ c) the king of spade? $\frac{1}{52}$

2. A coin is tossed twice. What is the probability of getting a total of

a) one tails? $\frac{1}{2}$ b) two tails? $\frac{1}{4}$ c) at least one tails? $\frac{3}{4}$

3. In a family with three children, calculate the probability of the following event:

a) the oldest child is a boy. $\frac{1}{2}$ b) there is a total of 3 boys. $\frac{1}{8}$

c) there is a total of at least one boy. $\frac{7}{8}$ d) there are more boys than girls. $\frac{1}{2}$

4. A die is rolled twice. Calculate the probability of the following event:

a) we get a "6" on the first roll. $\frac{1}{6}$

b) we get a "6" on the second roll. $\frac{1}{6}$

c) we get a "6" on the first or second roll. $\frac{11}{36}$

d) we get a "6" on each roll. $\frac{1}{36}$

e) we get no "6". $\frac{25}{36}$

5. A die is rolled twice. Calculate the probability of the following event:

a) the sum of the outcomes is equal to 10. $\frac{1}{12}$

b) the sum of the outcomes is greater than 10. $\frac{1}{12}$

c) the product of the outcome is equal to 12. $\frac{1}{9}$

d) we get the same outcome on each roll. $\frac{1}{6}$

e) the 1st outcome is even and the sum is equal to 7. $\frac{1}{12}$

6. A die is rolled and a coin is tossed at the same time. Calculate the probability of the following event:

a) the die shows "6". $\frac{1}{6}$

b) the coin shows "tails". $\frac{1}{2}$

c) the die shows "6" and the coin shows "tails". $\frac{1}{12}$

d) the die does not show "6" and the coin does not show "tails". $\frac{5}{12}$

e) the number on the die is even and the coin shows "tails". $\frac{1}{4}$

7. A coin is tossed three times. Calculate the probability of the following event:

a) we get tails on each toss. $\frac{1}{8}$

b) we get tails on the 3rd toss. $\frac{1}{2}$

c) we get a total of at least one tails. $\frac{7}{8}$

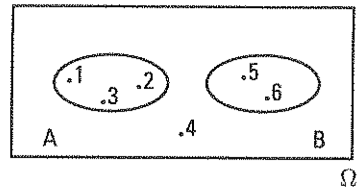
d) we get tails on the 1st and 3rd tosses. $\frac{1}{4}$

e) we get tails on the 1st or 3rd toss. $\frac{3}{4}$

5.6 Axioms and theorems

ACTIVITY 1 Axioms for calculating probabilities

A die is rolled once and the 6 possible outcomes are **equally likely**.
Consider the following incompatible events,
A: "rolling a number less than 4" and
B: "rolling a number greater than 4".
Using events A and B, verify the following three axioms.



Axiom 1: For any event E, we have: $0 \leq P(E) \leq 1$.

$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$; $0 \leq P(A) \leq 1$; $0 \leq P(B) \leq 1$

Axiom 2: $P(\Omega) = 1$

$P(\Omega) = \frac{n(\Omega)}{n(\Omega)} = \frac{6}{6} = 1$

Axiom 3: If A and B are two incompatible events, $P(A \cup B) = P(A) + P(B)$.

$A \cup B = \{1, 2, 3, 5, 6\}$; $P(A \cup B) = \frac{5}{6}$; $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$.

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$

These three axioms remain valid when the possible outcomes of a random experiment are not equally likely.

AXIOMS FOR CALCULATING PROBABILITIES

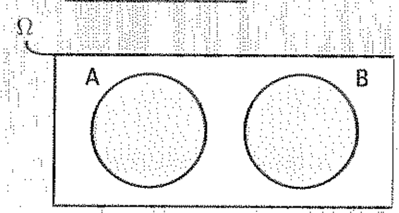
Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be the universal set associated with a random experiment where the possible outcomes $\omega_1, \omega_2, \ldots, \omega_n$ are not necessarily equally likely.

Axiom 1: For any event E, we have: $0 \leq P(E) \leq 1$

Axiom 2: The probability that the certain event occurs is equal to 1. $P(\Omega) = 1$

Axiom 3: If A and B are two mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$



Axiom 4: If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, we have:

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

As a consequence of these axioms:

For any outcome ω, the probability $P(\omega)$ is positive. $P(\omega) > 0$

The sum of the probabilities of the possible outcomes is equal to 1. $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1$

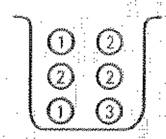
A consequence of these axioms is the following theorem which enables us to calculate the probability that an event occurs when the possible outcomes are not necessarily equally likely.

PROBABILITY OF AN EVENT

If A is any event, the probability P(A) that event A occurs is equal to the sum of the individual probabilities of the elements in A.

$$\text{If } A = \{a_1, a_2, \ldots, a_m\} \text{ then } P(A) = P(a_1) + P(a_2) + \ldots + P(a_m)$$

Ex.: Consider the jar on the right. A ball is drawn at random from the jar and its number is noted.



We have $\Omega = \{1, 2, 3\}$; $P(1) = \frac{2}{6}$; $P(2) = \frac{3}{6}$; $P(3) = \frac{1}{6}$.

The possible outcomes 1, 2 and 3 are not equally likely.

Let A be the event "drawing an odd number".

We have: $A = \{1, 3\}$ and $P(A) = P(1) + P(3) = \frac{3}{6}$.

1. Three horses H_1, H_3 and H_3 compete in a race. H_1 has twice as many chances as H_2 of winning and H_2 has twice as many chances as H_3 of winning. The experiment consists of observing the winner of the race.

a) Describe extensively the universal set Ω. ***$\Omega = \{H_1, H_2, H_3\}$***

b) If x represents the probability that H_3 wins, express in terms of x

1. the probability that H_2 wins. ***$2x$***
2. the probability that H_1 wins. ***$4x$***

c) Determine the respective probabilities $P(H_1)$, $P(H_2)$ and $P(H_3)$ of each of the three horses winning.

$P(C_1) = \frac{4}{7}$; $P(C_2) = \frac{2}{7}$; $P(C_3) = \frac{1}{7}$

2. A die is loaded so that when it is rolled, each even number has twice as many chances of showing than each odd number.

a) Let x be the probability that a 1 is rolled.

1. Complete the table on the right.

ω	1	2	3	4	5	6
P(ω)	x	$2x$	x	$2x$	x	$2x$

2. Determine the value of x.

$P(1) + P(2) + \ldots + P(6) = 1 \Leftrightarrow 9x = 1 \Leftrightarrow x = \frac{1}{9}$

3. Determine the probability of each outcome.

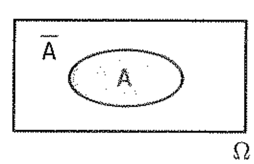
ω	1	2	3	4	5	6
P(ω)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

b) Calculate the probability of the following events.

1. "Rolling an even number." ***$\frac{2}{3}$***
2. "Rolling a number less than 4." ***$\frac{4}{9}$***

ACTIVITY 2 Probability of the contrary event

a) A die is rolled once. Consider the event A: "rolling a number greater than or equal to 5".



1. Describe in words the event $\overline{A}$.

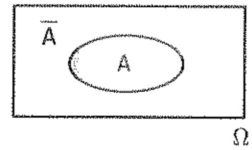
"Rolling a number less than 5."

2. After describing extensively the events A and $\overline{A}$ calculate

1) P(A). ***A = {5, 6}; P(A) = $\frac{1}{3}$*** 2) P($\overline{A}$). ***$\overline{A}$ = {1, 2, 3, 4}; P($\overline{A}$) = $\frac{2}{3}$***

3. Verify that $P(\overline{A}) = 1 - P(A)$. ***$\frac{2}{3} = 1 - \frac{1}{3}$***

b) Justify the steps showing that $P(\overline{A}) = 1 - P(A)$.

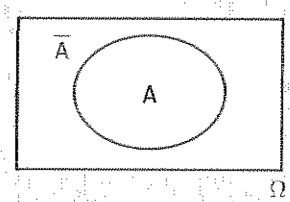


Statements	Justifications
1. $P(A \cup \overline{A}) = P(\Omega)$	***$A \cup \overline{A} = \Omega$***
2. $P(A \cup \overline{A}) = P(A) + P(\overline{A})$	***Axiom 3 is applied to the incompatible events A and $\overline{A}$.***
3. $P(\Omega) = 1$	***Axiom 2***
4. $P(A) + P(\overline{A}) = 1$	***Consequence of steps 1, 2 and 3***
5. $P(\overline{A}) = 1 - P(A)$	***Isolate $P(\overline{A})$.***

PROBABILITY OF THE CONTRARY EVENT

For any event A, we have

$$P(\overline{A}) = 1 - P(A)$$



Ex.: Let $\Omega = \{1, 2, ..., 6\}$ be the universal set associated with the experiment consisting of rolling a fair die once.

Let A: "rolling a number less than or equal to 2" $A = \{1, 2\}$; $P(A) = \frac{2}{6}$

and $\overline{A}$: "rolling a number greater than 2". $\overline{A} = \{3, 4, 5, 6\}$; $P(\overline{A}) = \frac{4}{6}$

We have: $P(\overline{A}) = 1 - P(A)$.

ACTIVITY 3 Probability of the impossible event

Justify the steps showing that $P(\varnothing) = 0$.

Statements	Justifications
1. $\varnothing = \overline{\Omega}$	***The impossible event is the contrary of the certain event.***
2. $P(\overline{\Omega}) = 1 - P(\Omega)$	***Theorem: For any event A, $P(\overline{A}) = 1 - P(A)$***
3. $P(\overline{\Omega}) = 1 - 1 = 0$	***$P(\Omega) = 1$ (Axiom 2)***
4. $P(\varnothing) = 0$	***$\varnothing = \overline{\Omega}$ (Step 1)***

PROBABILITY OF THE IMPOSSIBLE EVENT

The probability that the impossible event occurs is zero.

$$P(\varnothing) = 0$$

ACTIVITY 4 Probability of the event "A or B"

Consider the random experiment consisting of rolling a die once.
Let A and B be the events A: "rolling a number greater than or equal to 5" and B: "rolling an even number".

a) Describe extensively the following events.

1. $A =$ *{5, 6}*
2. $B =$ *{2, 4, 6}*
3. $A \cap B =$ *{6}*
4. $A \cup B =$ *{2, 4, 5, 6}*

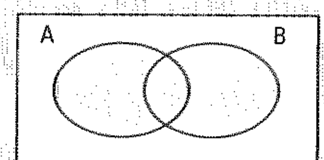
b) Calculate the following probabilities.

1. $P(A) = \frac{2}{6}$
2. $P(B) = \frac{3}{6}$
3. $P(A \cap B) = \frac{1}{6}$
4. $P(A \cup B) = \frac{4}{6}$

c) Verify that $P(A \cup B) = (P(A) + P(B) - P(A \cap B)$. $\frac{4}{6} = \frac{2}{6} + \frac{3}{6} - \frac{1}{6}$

PROBABILITY OF THE UNION

- Let A and B be two events associated with a random experiment. We have:



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex.: A card is drawn from a 52-card deck. Calculate the probability of drawing a queen or a heart.
Let A: "drawing a queen" and B: "drawing a heart".
We have $A \cup B$: "drawing a queen or a heart" and $A \cap B$: "drawing the queen of hearts".

$$P(A) = \frac{4}{52},\ P(B) = \frac{13}{52},\ P(A \cap B) = \frac{1}{52}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

ACTIVITY 5 Probability of the difference event

Consider the random experiment consisting of rolling a die once.
Let A and B be the events A: "rolling an even number" and B: "rolling a number less than 5".

a) Describe extensively the following events.

1. $A =$ *{2, 4, 6}*
2. $B =$ *{1, 2, 3, 4}*
3. $A \cap B =$ *{2, 4}*
4. $A \cap \overline{B} =$ *{6}*
5. $P(A \cap \overline{B}) =$ *{1, 3}*

b) Calculate the following probabilities.

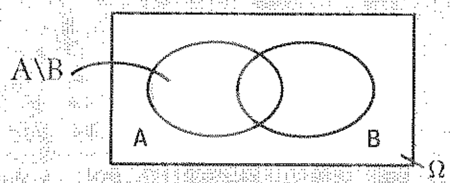
1. $P(A) = \frac{3}{6}$ 2. $P(B) = \frac{4}{6}$ 3. $P(A \cap B) = \frac{2}{6}$ 4. $P(A \cap \overline{B}) = \frac{1}{6}$ 5. $P(\overline{A} \cap B) = \frac{2}{6}$

c) Verify that

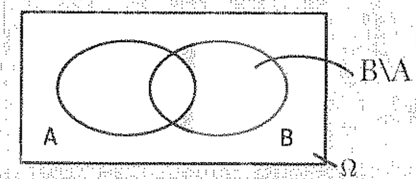
1. $P(A \cap \overline{B}) = P(A) - P(A \cap B)$. $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$
2. $P(\overline{A} \cap B) = P(B) - P(A \cap B)$. $\frac{1}{3} = \frac{2}{3} - \frac{1}{3}$

PROBABILITY OF THE DIFFERENCE

- Let A and B be two events associated with a random experiment. We have:



$P(A \setminus B) = P(A) - P(A \cap B)$



$P(B \setminus A) = P(B) - P(A \cap B)$

Ex.: A card is drawn from a 52-card deck.

Let A and B be the events A: "drawing a queen" and B: " drawing a heart".

$P(A) = \frac{4}{52}$; $P(B) = \frac{13}{52}$; $P(A \cap B) = \frac{1}{52}$

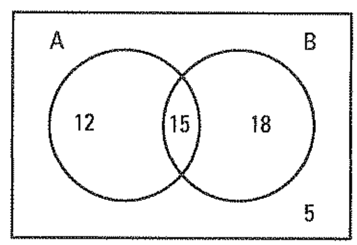
A\B: "drawing a queen that is not a heart".

B\A: "drawing a heart that is not a queen".

$P(A \setminus B) = P(A) - P(A \cap B) = \frac{3}{52}$; $P(B \setminus A) = P(B) - P(A \cap B) = \frac{12}{52}$

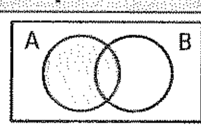
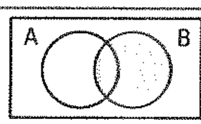
3. A study on a group of 50 students who wrote English and biology exams shows that

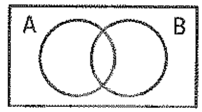
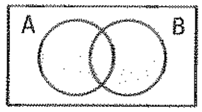
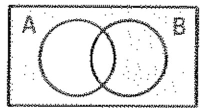
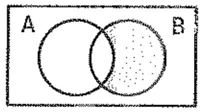
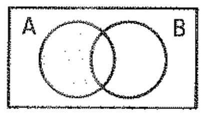
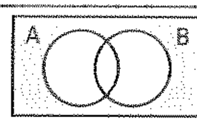
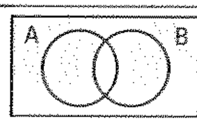
- 15 students passed both exams;
- 12 students passed the English exam but failed the biology one;
- 18 students passed the biology exam but failed the English one;
- 5 failed both exams.



A student is chosen at random from the group. Consider the following events:
A: "passing the English exam"; B: "passing the biology exam".

a) Complete the following table.

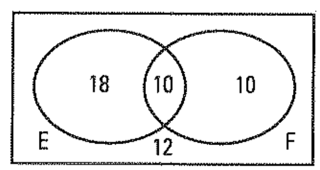
	Event	Representation	Interpretation	Probability
1.	A		*"passing the English exam"*	*P(A) = 0.54*
2.	B		*"passing the biology exam"*	*P(B) = 0.66*

3.	$A \cap B$		***"passing the English and biology exams"***	***$P(A \cap B) = 0.30$***
4.	$A \cup B$		***"passing the English or the biology exam"***	***$P(A \cup B) = 0.90$***
5.	$\overline{A}$		***"failing the English exam"***	***$P(\overline{A}) = 0.46$***
6.	$\overline{A} \cap B$		***"passing the biology exam but failing the English exam"***	***$P(\overline{A} \cap B) = 0.36$***
7.	$A \cap \overline{B}$		***"passing the English exam but failing the biology exam"***	***$P(A \cap \overline{B}) = 0.24$***
8.	$\overline{A} \cap \overline{B} = \overline{A \cup B}$		***"failing both exams"***	***$P(\overline{A} \cap \overline{B}) = 0.10$***
9.	$\overline{A} \cup \overline{B} = \overline{A \cap B}$		***"failing one of the exams"***	***$P(\overline{A} \cup \overline{B}) = 0.70$***

b) Verify that

1. $P(\overline{A}) = 1 - P(A)$. ***0.46 = 1 - 0.54***
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. ***0.90 = 0.54 + 0.66 - 0.30***
3. $P(A \cap \overline{B}) = P(A) - P(A \cap B)$. ***0.24 = 0.54 - 0.30***
4. $P(\overline{A} \cap B) = P(B) - P(A \cap B)$. ***0.36 = 0.66 - 0.30***
5. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$. ***0.10 = 1 - 0.90***
6. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$. ***0.70 = 1 - 0.30***

4. A study shows that, in a group of 50 tourists, 28 speak English, 20 speak French and 10 speak English and French. A tourist is chosen at random from the group. Consider the following events:
E: "speaking English"; F: "speaking French".



a) Represent the situation on the Venn diagram on the right.

b) Write symbolically the following events and then calculate the probability of each of them.

1. The tourist speaks English. ***$P(E) = 0.56$***
2. The tourist speaks English but not French. ***$P(E \cap \overline{F}) = 0.36$***
3. The tourist speaks English and French. ***$P(E \cap F) = 0.20$***
4. The tourist speaks English or French. ***$P(E \cup F) = 0.76$***
5. The tourist speaks neither English nor French. ***$P(\overline{E} \cap \overline{F}) = P(\overline{E \cup F}) = 0.24$***
6. The tourist does not speak English or does not speak French. ***$P(\overline{E} \cup \overline{F}) = P(\overline{E \cap F}) = 0.80$***

5. Let A and B be two events such that $P(A) = 0.7$; $P(B) = 0.6$; $P(A \cap B) = 0.5$. Calculate the following probabilities.

a) $P(\overline{A}) =$ ***$1 - P(A) = 0.3$***

b) $P(A \cup B) =$ ***$P(A) + P(B) - P(A \cap B) = 0.8$***

c) $P(A \cap \overline{B}) =$ ***$P(A) - P(A \cap B) = 0.2$***

d) $P(\overline{A \cup B}) =$ ***$1 - P(A \cup B) = 0.2$***

e) $P(\overline{A} \cup \overline{B}) =$ ***$P(\overline{A \cap B}) = 1 - P(A \cap B) = 0.5$***

f) $P(\overline{A \cap B}) =$ ***$1 - P(A \cap B) = 0.5$***

g) $P(\overline{A} \cap \overline{B}) =$ ***$P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.2$***

h) $P(\overline{A} \cap \overline{B}) =$ ***$P(A) + P(\overline{B}) - P(A \cap \overline{B}) = 0.9$***

6. A study on drivers summoned to the municipal court for traffic law violations shows that

- 75% of the drivers are actually guilty;
- 60% of the drivers are convicted for the violation;
- 50% of the drivers are actually guilty and convicted for the violation.

A driver appears before the judge. Consider the following events:
A: "the driver is actually guilty"; B: "the driver is convicted for the violation".

Calculate and interpret the following probabilities:

a) $P(\overline{A})$ ***$1 - P(A) = 0.25$. There is a 25 out of 100 chance that the driver is not guilty.***

b) $P(A \cup B)$ ***$P(A) + P(B) - P(A \cap B) = 0.85$. There is a 85 out of 100 chance that the driver is actually guilty or is convicted.***

c) $P(A \cap \overline{B})$ ***$P(A) - P(A \cap B) = 0.25$. There is a 25 out of 100 chance that the driver is actually guilty and is not convicted.***

d) $P(\overline{A} \cap \overline{B})$ ***$P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.15$. There is a 15 out of 100 chance that the driver is not guilty and is not convicted.***

e) $P(\overline{A} \cup \overline{B})$ ***$P(\overline{A \cap B}) = 1 - P(A \cap B) = 0.50$. There is a 50 out of 100 chance that the driver is not guilty or is not convicted.***

7. An economist makes the following predictions for next week.

- There is a 40 out of 100 chance that the dollar will fall.
- There is a 30 out of 100 chance that interest rates will fall.
- There is a 20 out of 100 chance that both the dollar and interest rates will fall.

Calculate the probability that next week

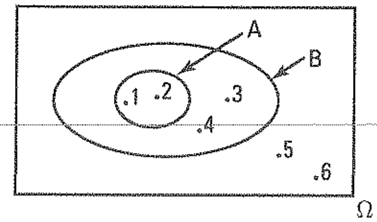
a) the dollar or interest rates will fall. ***0.50***

b) the dollar will fall and interest rates will not fall. ***0.20***

c) the dollar and interest rates will not fall. ***0.50***

d) the dollar or interest rates will not fall. ***0.80***

8. A die is rolled once. Consider the following events.
A: " rolling a number less than 3" and
B: "rolling a number less than 5".



a) Describe extensively events A and B, represent them on the Venn diagram on the right and verify that $A \subseteq B$.

A = {1, 2}; B = {1, 2, 3, 4}

b) If event A occurs, can you conclude that event B occurs? *Yes*

c) What can you conclude concerning the occurrence of event A when you know that event B does not occur?

A does not occur either.

d) Verify the property: $A \subseteq B \Rightarrow P(A) \leqslant P(B)$.

$P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{3}$. Indeed, we have $A \subseteq B$ and $P(A) \leqslant P(B)$.

e) Calculate

1. $P(A \cap B)$. *$P(A) = \frac{1}{3}$*
2. $P(A \cup B)$. *$P(B) = \frac{1}{2}$*

9. A die is rolled twice. Calculate the probability of the following events.

a) "We get a 6 on the 1st or 2nd roll". *$\frac{11}{36}$*

b) "The sum of the numbers is equal to 7". *$\frac{1}{6}$*

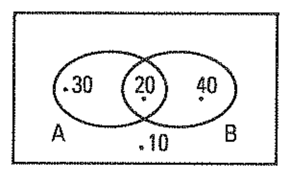
c) "The sum of the numbers is equal to 7 or we get an even number on the 1st roll." *$\frac{7}{12}$*

d) "We get the same number on each roll". *$\frac{1}{6}$*

e) "The sum of the numbers is equal to 6 and we get a 6 on the 1st roll". *0*

f) "We get an even or an odd number on the 1st roll". *1*

10. A meteorologist makes the following forecast for tomorrow.
- There is a 50 out of 100 chance that the temperature will be below 0 °C.
- There is a 60 out of 100 chance that it will snow.
- There is a 30 out of 100 chance that the temperature will drop below 0 °C and that it will not snow.



A: "the temperature will be below 0°"
B: "it will snow"

What is the probability that tomorrow

a) the temperature is below 0 °C and it snows? *20 %*

b) it snows and the temperature is not below 0 °C? *40 %*

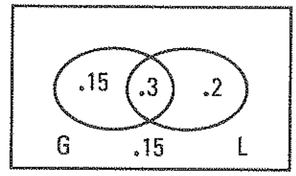
c) it does not snow and the temperature is not below 0 °C? *10 %*

d) the temperature is below 0 °C or it snows? *90 %*

5.7 Conditional probability

ACTIVITY 1 Conditional probability (equally likely outcomes situation)

In a group of 30 students, there are 18 boys. We observe a total of 3 boys and 2 girls wearing glasses. A student is chosen at random from the group.



Ω

B: "the student is a boy"

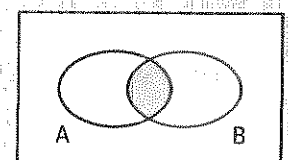
G: "the student wears glasses"

a) If the student is a boy, what is the probability that he wears glasses? $\frac{1}{6}$

b) If the chosen student wears glasses, what is the probability that it is a boy? $\frac{3}{5}$

CONDITIONAL PROBABILITY (equally likely outcomes situation)

Let Ω be the universal set associated with a random experiment and B an event such that all outcomes favourable to B are equally likely. The probability that an event A occurs given that the event B has occurred, written P(A|B), is defined by

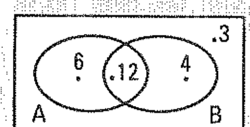


Ω

$$P(A \mid B) = \frac{\text{number of outcomes favourable to A among those favourable to B}}{\text{number of outcomes favourable to B}} = \frac{n(A \cap B)}{n(B)}$$

Ex.: In a group of 25 students, we observe that

- 18 pass the English test;
- 16 pass the biology test;
- 12 pass both tests.



Ω

The probability that a student passes in English if he passes in biology is written P(A|B).

We have: $P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{12}{16} = 75\ \%$.

The probability that a student passes in biology if he passes in English is written P(B|A).

We have $P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{12}{18} = 66.\bar{6}\ \%$.

1. Canadian tourists coming back from South America were interviewed. Among the 40 tourists in the group

- 20 tourists visited Argentina;
- 30 tourists visited Brazil;
- 12 tourists visited both these countries.

What is the probability that a tourist visited

a) Argentina if he visited Brazil? $\frac{2}{5}$

b) Brazil if he visited Argentina? $\frac{3}{5}$

c) Argentina if he did not visit Brazil? $\frac{4}{5}$

d) Brazil if he did not visit Argentina? $\frac{9}{10}$

2. The human resources director of a large company sorts 1000 employee files according to age and gender.

Age \ Gender	Male (M)	Female (F)	Total
under 30 years old (A)	100	150	250
30 to 40 years old (B)	240	210	450
over 40 years old (C)	180	120	300
Total	520	480	1000

a) A file is selected at random. What is the probability that the employee

1. is a male? ***0.52***
2. is under 30 years old? ***0.25***
3. is a male under 30? ***0.10***

b) What is the probability that the file is that of an employee who

1. is under 30 years old, knowing that the employee is a male? $P(A \mid M) = \frac{100}{520} \approx 0.192$
2. is a male, knowing that the employee is under 30? $P(M \mid A) = \frac{100}{250} = 0.40$

c) Interpret the following expressions and then calculate the probabilities.

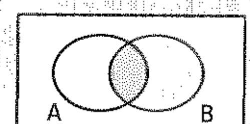
1. $P(F)$ ***Probability that the employee is a female: 0.48.***
2. $P(F|A)$ ***Probability that the employee is a female if the employee is under 30: 0.6.***
3. $P(A|F)$ ***Probability that the employee is under 30 if the employee is a female: 0.3125.***
4. $P(M|C)$ ***Probability that the employee is a male if the employee is older than 40: 0.6.***
5. $P(F|A \cup B)$ ***Probability that the employee is a female if the employee is 40 or younger: 0.514.***
6. $P(M|B \cup C)$ ***Probability that the employee is a male if the employee is 30 or older: 0.56.***

CONDITIONAL PROBABILITY

In general, given two events A and B,

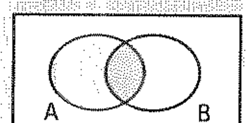
– the probability of A, knowing B, is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (P(B) \neq 0)$$



– the probability of B, knowing A, is defined by:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad (P(A) \neq 0)$$



Ex.: At a college,

– 60% of students pass the English exam ($P(A) = 0.6$);
– 50% of students pass the biology exam ($P(B) = 0.5$);
– 40% of students pass both exams ($P(A \cap B) = 0.40$).

The probability that a student passes in English, if he passed in biology, is equal to:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.40}{0.50} = \frac{4}{5}.$$

The probability that a student passes in biology, if he passed in English, is equal to:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.40}{0.60} = \frac{2}{3}.$$

3. A fair die is rolled once. Consider the following events.
A: "rolling an even number" and B: "rolling a number less than 5".

a) Calculate.

1. $P(A)$ $\frac{1}{2}$ 2. $P(B)$ $\frac{2}{3}$ 3. $P(A \cap B)$ $\frac{1}{3}$

4. $P(A|B)$ $\frac{1}{2}$ 5. $P(B|A)$ $\frac{2}{3}$

b) Verify.

1. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\frac{1}{2}$ 2. $P(B|A) = \frac{P(A \cap B)}{P(A)}$ $\frac{2}{3}$

4. For their extracurricular activities, 20% of the students at a school sing in the choir, 15% do theatre and 5% participate in both these activities. What is the probability that a student

a) sings in the choir if he does theatre? $\frac{1}{3}$

b) does theatre if he sings in the choir? $\frac{1}{4}$

5. A study done on secondary 5 students shows that
- 70% of students pass their mathematics course;
- 60% of students pass their physics course;
- 50% of students pass both courses.

A student is chosen at random. Consider the following events.
M: "the student passes in mathematics"; P: "the student passes in physics".
Calculate the following probabilities and interpret the results.

a) $P(M|P) = \frac{5}{6}$ *If the student passes in physics, he has a five out of six chance of passing in mathematics.*

b) $P(\overline{M}\,|\,P) = \frac{1}{6}$ *If the student passes in physics, he has a one out of six chance of failing in mathematics.*

c) $P(M\,|\,\overline{P}) = 0.5$ *If the student fails in physics, he has a one out of two chance of passing in mathematics.*

d) $P(\overline{M}\,|\,\overline{P}) = 0.5$ *If the student fails in physics, he has a one out of two chance of failing in mathematics.*

e) $P(P\,|\,M) = \frac{5}{7}$ *If the student passes in mathematics, he has a five out of seven chance of passing in physics.*

f) $P(\overline{P}\,|\,M) = \frac{2}{7}$ *If the student passes in mathematics, he has a two out of seven chance of failing in physics.*

g) $P(\overline{P}\,|\,\overline{M}) = \frac{2}{3}$ *If the student fails in mathematics, he has a two out of three chance of failing in physics.*

h) $P(P\,|\,\overline{M}) = \frac{1}{3}$ *If the student fails in mathematics, he has a one out of three chance of passing in physics.*

6. In a survey, 40% of people said they read weekly magazines, 50% read monthly magazines and 20% reads both types of magazines. What is the probability that one of the surveyed individual

a) reads monthly magazines if he reads weekly magazines? *0.50*

b) reads weekly magazines if he reads monthly magazines? *0.40*

7. a) In a random experiment, two events A and B such that $A \subseteq B$ are considered. What can be said about $P(B|A)$?

Justify your answer. ***$P(B|A) = 1$. Indeed, $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)}$***

b) A card is drawn from a 52-card deck. Consider the events A: "drawing a queen" and B: "drawing a face card".

1. What can be said about the events A and B? ***$A \subseteq B$***
2. Calculate $P(B|A)$. ***$= 1$***
3. Calculate $P(A|B)$. ***$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}$***

Activity 2 Probability of the event "A and B"

a) Let A and B be any two events associated with a random experiment and such that $P(A) \neq 0$ and $P(B) \neq 0$. Show that

1. $P(A \cap B) = P(A) \times P(B|A)$.

   We have: $P(B|A) = \frac{P(B \cap A)}{P(A)}$. We deduce that $P(B \cap A) = P(A) \times P(B|A)$.

2. $P(A \cap B) = P(B) \times P(A|B)$.

   We have: $P(A|B) = \frac{P(A \cap B)}{P(B)}$. We deduce that $P(A \cap B) = P(B) \times P(A|B)$.

b) In a company, 40% of employees are men and 20% of men are Anglophone. An employee is chosen at random. Consider the following events:

M: "the employee is a man", A: "the employee is Anglophone".

Calculate the probability that the chosen employee is an Anglophone man.

$P(H \cap A) = P(H) \times P(A|H) = 0.40 \times 0.20 = 0.08$

PROBABILITY OF THE INTERSECTION

Let A and B be two events associated with a random experiment and such that $P(A) \neq 0$ and $P(B) \neq 0$.

We have: $\boxed{P(A \cap B) = P(A) \times P(B|A)}$ or $\boxed{P(A \cap B) = P(B) \times P(A|B)}$

Ex.: In a factory where bolts are made, a machine M provides 40% of the total production. A study showed that 10 % of the bolts produced by machine M are defective. A bolt is chosen at random from the total production. What is the probability that the bolt comes from machine M and is defective?

Consider the events M: "the bolt comes from machine M" and D: "the bolt is defective".

We have: $P(M) = 0.40$; $P(D|M) = 0.10$. We seek $P(M \cap D)$.

$P(M \cap D) = P(M) \times P(D|M) = 0.40 \times 0.10 = 0.04$

8. In a class, 40% of students are boys. In addition, 20% of boys have blue eyes while 25% of girls have blue eyes. A student is chosen at random in this class. What is the probability that the chosen student is

a) a boy with blue eyes? ***0.08*** **b)** a girl with blue eyes? ***0.15***

c) a boy who doesn't have blue eyes? ***0.32*** **d)** a girl who doesn't have blue eyes? ***0.45***

ACTIVITY 3 Tree

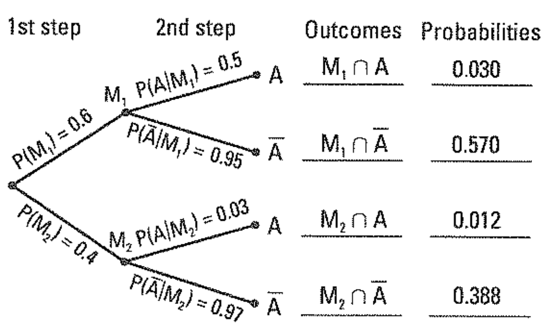
In a factory that makes bolts, production is done by two machines, M_1 and M_2.
M_1 provides 60% of the total production and M_2 provides 40 %.
Among the bolts produced by M_1, 5% are defective, while 3% of those produced by M_2 are.
A bolt is chosen at random from the total production.
First, its origin is examined: M_1 or M_2, then its state: defective (A) or non defective ($\bar{A}$).

a) Calculate the following probabilities and interpret the results.

1. $P(M_1)$ ***Probability that the bolt comes from machine M_1: 0.60.***
2. $P(M_2)$ ***Probability that the bolt comes from machine M_2: 0.40.***
3. $P(A|M_1)$ ***Probability that the bolt is defective if it comes from machine M_1: 0.05.***
4. $P(A|M_2)$ ***Probability that the bolt is defective if it comes from machine M_2: 0.03.***
5. $P(M_1 \cap A)$ ***Probability that the bolt comes from machine M_1 and is defective: 0.030.***
6. $P(M_2 \cap A)$ ***Probability that the bolt comes from machine M_2 and is defective: 0.012.***

b) Complete the tree below by indicating the probabilities on each branch, then determine the probability of each outcome. (Note: the outcome $M_1 \cap A$ can be written: (M_1, A)).

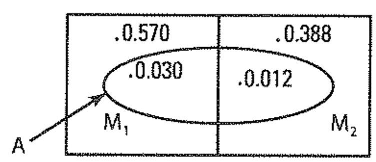
The 1st step consists of noting the origin of the bolt and the 2nd consists of indicating the state of the bolt.



c) Complete the extensive description of the set Ω of possible outcomes.
$\Omega = \{(M_1, A),$ ***$(M_1, \bar{A}), (M_2, A), (M_2, \bar{A})$*** $\}$

d) Verify that the sum of the probabilities of the possible outcomes is equal to 1.
0.030 + 0.570 + 0.012 + 0.388 = 1

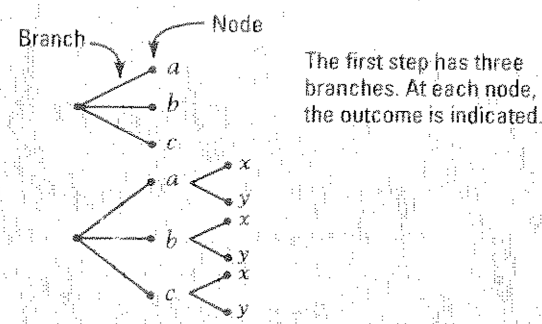
e) Write the probability of each possible outcome on the Venn diagram on the right.

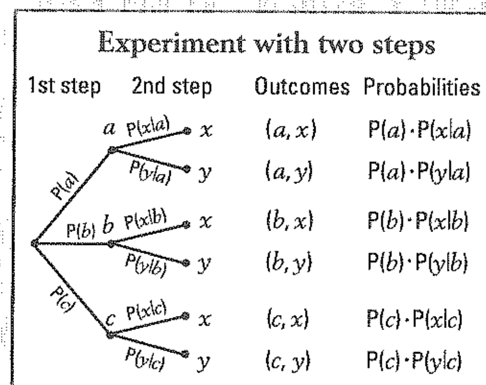


TREE

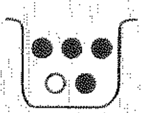
The outcomes of a random experiment having many steps can be determined using a tree or rooted tree diagram.

- For the first step, draw as many branches as there are possible outcomes for the first step.
- For the second step, draw, starting from each node of the first step, as many branches as there are possible outcomes for the second step.
- Indicate the possible outcomes of the experiment.
- On each branch, indicate the probability.
- Deduce the probability of each outcome by calculating the product of the probabilities indicated on the branches giving rise to this outcome.

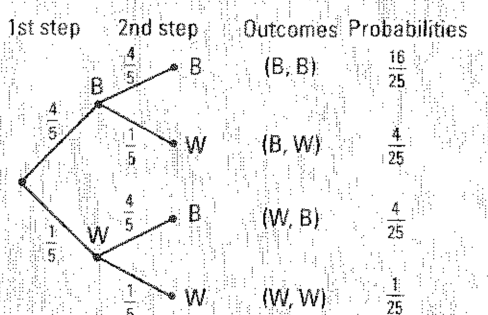




Ex.: Two balls are drawn successively from a jar containing four black balls and one white ball.

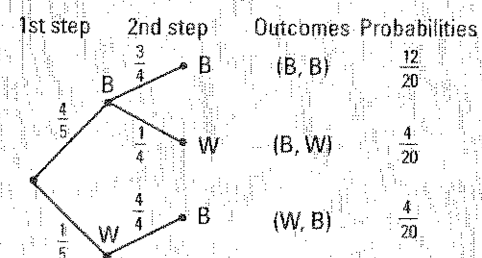


First situation: draw with replacement



$\Omega = \{(B, B), (B, W), (W, B), (W, W)\}$

Second situation: draw without replacement



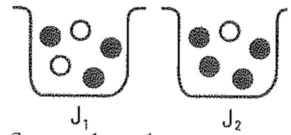
$\Omega = \{(B, B), (B, W), (W, B)\}$

The event E: "drawing a total of one white ball" is described extensively by $E = \{(W, B), (B, W)\}$.

The probability that event E occurs is

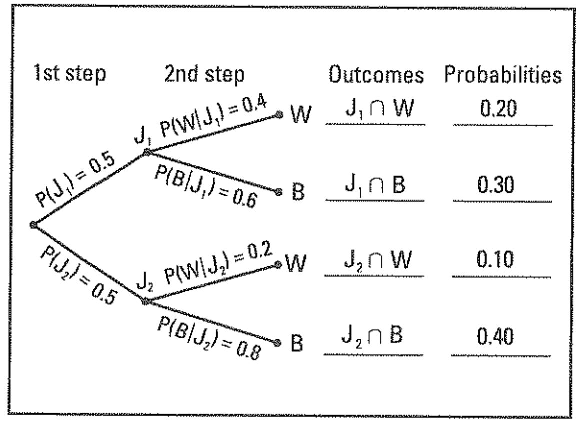
- draw with replacement: $P(E) = P(W, B) + P(B, W) = \frac{4}{25} + \frac{4}{25} = \frac{8}{25}$.
- draw without replacement: $P(E) = P(W, B) + P(B, W) = \frac{4}{20} + \frac{4}{20} = \frac{8}{20}$.

9. Consider two jars.



Jar J_1 contains three black balls and two white balls;
Jar J_2 contains four black balls and one white ball.
The experiment consists of choosing a jar at random, then drawing a ball from the chosen jar.

Consider the events:
J_1: "jar J_1 is chosen";
J_2: "jar J_2 is chosen";
B: "a black ball is drawn";
W: "a white ball is drawn".



a) Represent the situation using a tree and indicate the probabilities on each branch, then determine the probability of each outcome.

b) Determine the set Ω of possible outcomes.
$\Omega = \{(J_1, W), (J_1, B), (J_2, W), (J_2, B)\}$

c) Verify that the sum of the probabilities of the possible outcomes is equal to 1. ***$0.2 + 0.3 + 0.1 + 0.4 = 1$***

d) Consider the event W: "drawing a white ball".
1. Describe event W extensively. ***$W = \{(J_1, W), (J_2, W)\}$***
2. Calculate P(W). ***$P(W) = P(J_1, W) + P(J_2, W) = 0.30$***

e) Determine the probability that jar J_1 was chosen knowing that the ball drawn is white.
$$P(J_1|W) = \frac{P(J_1 \cap W)}{P(W)} = \frac{0.20}{0.30} = \frac{2}{3}$$

10. Consider the two jars from exercise 9.
The experiment consists of drawing a card from a 52-card deck. If the card drawn is a heart, a ball is drawn from jar J_1, otherwise a ball is drawn from jar J_2.

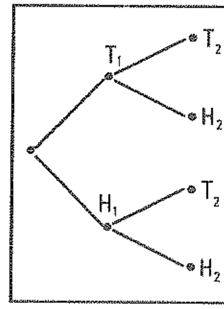
a) Determine the probability of each possible outcome.
$P(J_1, W) = 0.1$; $P(J_1, B) = 0.15$; $P(J_2, W) = 0.15$; $P(J_2, B) = 0.6$

b) Calculate the probability of the event W: "drawing a white ball".
$P(W) = P(J_1, W) + P(J_2, W) = 0.25$

c) Determine the probability that jar J_1 was chosen knowing that the ball drawn is white.
$P(J_1 | W) = 0.4$

5.8 Independent events

ACTIVITY 1 Independent events



A coin is tossed twice. Consider the following events.
T_i: "getting tails on the ith toss", $i = 1, 2$.
H_i: "getting heads on the ith toss", $i = 1, 2$.

a) Determine.

1. $P(T_1)$ ***$\frac{1}{2}$*** 2. $P(T_2)$ ***$\frac{1}{2}$***
3. $P(T_2 | T_1)$ ***$\frac{1}{2}$*** 4. $P(T_2 | H_1)$ ***$\frac{1}{2}$***

b) Answer true of false.

1. $P(T_2 | T_1) = P(T_2)$ ***True*** 2. $P(T_2 | H_1) = P(T_2)$ ***True***

c) Does the probability of getting tails on the 2nd toss depend on the outcome of the 1st toss?
No

d) 1. Describe in words the event $T_1 \cap T_2$ then calculate $P(T_1 \cap T_2)$.
$T_1 \cap T_2$: "getting tails on each toss"; $P(T_1 \cap T_2) = \frac{1}{4}$

2. Verify that $P(T_1 \cap T_2) = P(T_1) \times P(T_2)$. ***$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$***

INDEPENDENT EVENTS

- Two events A and B are independent if
  - the probability that event A occurs is not influenced by the fact that event B occurred.

$$P(A|B) = P(A)$$

  - the probability that event B occurs is not influenced by the fact that event A occurred.

$$P(B|A) = P(B)$$

- Two events A and B that are not independent are called dependent.
  Ex.: A jar contains red balls and black balls.
  - When two balls are drawn with replacement, the event R_2: "getting a red ball on the second draw" is independent from the event R_1: "getting a red ball on the first draw". We have: $P(R_2) = P(R_2 | R_1)$. The draws are called independent.
  - When two balls are drawn without replacement, events R_1 and R_2 are dependent. We have: $P(R_2) \neq P(R_2 | R_1)$. The draws are called dependent.
- **Theorem**: Let A and B be two events associated with a random experiment. We have:

$$\text{A and B are independent if, and only if, } P(A \cap B) = P(A) \times P(B)$$

Ex.: A die is rolled at the same time a coin is tossed.
The events A: "the die shows 6" and B: "the coin shows tails" are independent.
We have: $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.
Thus, there is a one out of 12 chance that the die shows 6 and the coin shows tails.

1. Complete using the appropriate symbol = or ≠.
Two events A and B are dependent if

1. $P(A|B)$ ≠ $P(A)$. 2. $P(B|A)$ ≠ $P(B)$. 3. $P(A \cap B)$ ≠ $P(A) \times P(B)$.

2. A die is rolled twice. Consider the following events.
A: "getting 6 on the 1st roll", B: "getting 6 on the 2nd roll" and C: "rolling a sum equal to 10".

a) Are events A and B independent? Justify your answer.

$P(A) = \frac{1}{6}$; $P(B) = \frac{1}{6}$; $P(A \cap B) = \frac{1}{36}$

Since $P(A \cap B) = P(A) \times P(B)$, we conclude that events A and B are independent.

b) Are events A and C dependent? Justify your answer.

$P(A) = \frac{1}{6}$; $P(C) = \frac{3}{36}$; $P(A \cap C) = \frac{1}{36}$

Since $P(A \cap C) \neq P(A) \times P(C)$, we conclude that events A and C are dependent.

3. Consider, in a family with 3 children, the following events.
A: "the oldest child is a boy", B: "all the children are of same gender" and C: "there is a total of two boys in the family".

a) Determine if events A and B are independent.

A = {(B, G, G), (B, G, B), (B, B, G), (B, B, B)}; B = {(B, B, B), (G, G, G)}

$A \cap B$ = {(B, B, B)}; $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{8}$

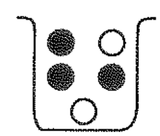
Since $P(A \cap B) = P(A) \times P(B)$, events A and B are independent.

b) Determine if events A and C are independent.

C = {(B, B, G), (B, G, B), (G, B, B)}; $A \cap C$ = {(B, B, G), (B, G, B)}

$P(C) = \frac{3}{8}$, $P(A \cap C) = \frac{1}{4}$. Since $P(A \cap C) \neq P(A) \times P(C)$, events A and C are dependent.

4. A jar contains 2 white balls and 3 black balls.
Three balls are drawn successively **without replacement**.
Consider the following events.
W_i: "getting a white ball on the *i*th draw", $i = 1, 2, 3$.
B_i: "getting a black ball on the *i*th draw", $i = 1, 2, 3$.
Calculate and interpret the outcomes



a) $P(B_1)$ ***$\frac{2}{5}$. There is a 2 out of 5 chance of getting a white ball on the 1st draw.***

b) $P(B_2|B_1)$ ***$\frac{1}{4}$. There is a 1 out of 4 chance of getting a white ball on the 2nd draw if the first ball drawn is white.***

c) $P(B_3|B_1 \cap B_2)$ ***0. It is impossible to get a white ball on the 3rd draw if a white ball was drawn on the first two draws.***

d) $P(B_1 \cap B_2)$ ***$\frac{1}{10}$. There is a 1 out of 10 chance of getting a white ball on the 1st and 2nd draws.***

5. Three balls are drawn successively **with replacement** from the jar of exercise 4. Calculate.

a) $P(W_1)$ $\frac{2}{5}$

b) $P(W_2|W_1)$ $\frac{2}{5}$

c) $P(W_1 \cap W_2)$ $\frac{4}{25}$

d) $P(W_3|W_1 \cap W_2)$ $\frac{2}{5}$

e) $P(W_1 \cap W_2 \cap W_3)$ $\frac{8}{125}$

SEQUENCE OF INDEPENDENT EVENTS

Let $A_1, A_2, \ldots, A_n$, be n events associated with a random experiment.

$A_1, A_2, \ldots, A_n$ are independent if, and only if, $P(A_1 \cap A_2 \cap \ldots A_n) = P(A_1) \times P(A_2) \times \ldots P(A_n)$

Ex.: A die is rolled three times. Consider the following events.

A_i: "getting a 6 on the *i*th roll", $i = 1, 2, 3$.

– A_1, A_2 and A_3 are independent and $P(A_i) = \frac{1}{6}$

– $P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

– There is a one out of 216 chance of getting a 6 on each roll.

6. At archery, Riva hits the target 3 times out of 5. She shoots three times at the target. Calculate the probability of the following events.

a) She hits the target on the 1st shot, misses on the second and hits the target on the third.
$\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{125}$

b) She hits the target on each shot. $\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

c) She misses the target on each shot. $\frac{8}{125}$

d) She hits the target only once. $\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{36}{125}$

7. Four cards are drawn successively from a 52-card deck. Calculate the probability of getting, on each draw, a heart when the draw is done

a) with replacement. $\frac{13}{52} \times \frac{13}{52} \times \frac{13}{52} \times \frac{13}{52} = \frac{1}{256}$

b) without replacement. $\frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} = \frac{11}{4165}$

8. Nathalie and Eric write a mathematics exam. Nathalie has a 7 out of 10 chance to pass the exam while Eric has a 6 out of 10 chance.

Calculate the probability of the following events.

a) "Both Nathalie and Eric pass." $0.7 \times 0.6 = 0.42$

b) "Nathalie passes and Eric fails." $0.7 \times 0.4 = 0.28$

c) "Nathalie fails and Eric passes." $0.3 \times 0.6 = 0.18$

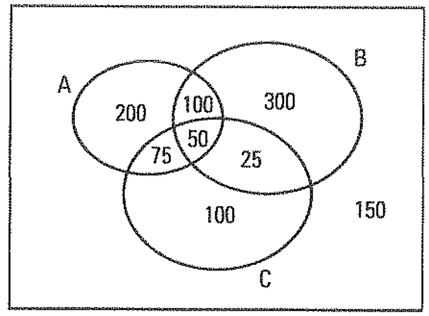
d) "Both Nathalie and Eric fail." $0.3 \times 0.4 = 0.12$

e) "Only one of the two passes." $0.7 \times 0.4 + 0.3 \times 0.6 = 0.46$

f) "At least one of the two passes." $0.3 \times 0.6 + 0.7 \times 0.4 + 0.7 \times 0.6 = 0.88$ or $1 - 0.12 = 0.88$

5.9 Problems

1. A survey of 1000 people is done regarding magazines A, B and C. The survey shows that 425 people read magazine A; 475 read magazine B; 250 read magazine C; 150 read magazines A and B; 125 read magazines A and C; 75 read magazines B and C; 50 read magazines A, B and C. Calculate the probability that a person in this survey reads



a) two of these magazines. ***0.20***

b) none of these magazines. ***0.15***

2. A and B are two events such that $P(A) = 0.4$ and $P(A \cup B) = 0.6$. Détermine $P(B)$ if

a) A and B are incompatible. $P(A \cup B) = P(A) + P(B) \Rightarrow P(B) = 0.2$

b) A and B are independent. $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B) \Rightarrow P(B) = \frac{1}{3}$

c) event A implies event B. $P(A \cap B) = P(A) \Rightarrow P(A \cup B) = P(B) \Rightarrow P(B) = 0.6$

3. Consider two jars. Jar J_1 contains three red balls and two white balls; jar J_2 contains four red balls and one white ball. A jar is chosen at random and a ball is drawn from it.

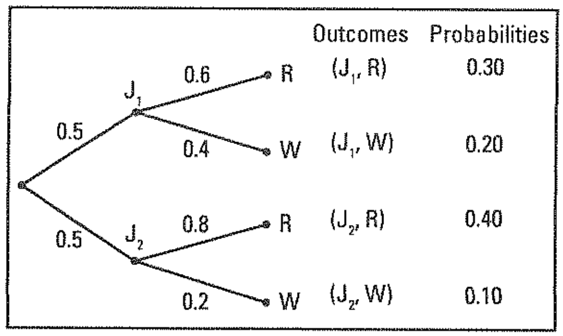
a) Calculate the probability of drawing a red ball.

$R = \{(J_1, R), (J_2, R)\}$

$P(R) = 0.30 + 0.40 = 0.70$

b) If the ball drawn is red, compute the probability that it came from jar J_1.

$P(J_1 \mid R) = \frac{P(J_1 \cap R)}{P(R)} = \frac{0.30}{0.70} = \frac{3}{7}$



4. A die is rolled twice. Calculate the following probabilities:

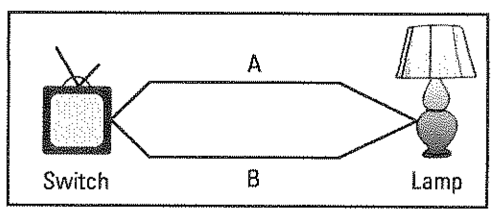
a) "Getting a 6 on the first or second roll." $\frac{11}{36}$

b) "Getting a 6 on the second roll if we got a 6 on the first roll." $\frac{1}{6}$

c) "Getting a sum equal to 7 if we got a 6 on the first roll." $\frac{1}{6}$

d) "Getting a 6 on the first roll if the sum is equal to 7." $\frac{1}{6}$

5. In the electric circuit on the right, cables A and B work independently. Cable A has a 9 out of 10 chance of working, while cable B has an 8 out of 10 chance. When the switch is activated, the lamp does not turn on if both cables are malfunctioning. The switch is activated. What is the probability that the lamp will turn on?



A: "cable A is working", B: "cable B is working", A and B are independent.

$A \cup B$: "the light is turned on". $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9 + 0.8 - 0.9 \times 0.8 = 0.98$

6. A die is rolled three times. What is the probability that the outcome of the first roll is an odd number, the outcome of the second roll is an even number and the outcome of the third roll is greater than 3?

$\frac{1}{12}$

7. The hockey and football teams of a college play on the same day. The probability that the hockey team wins is $\frac{7}{15}$ and the probability that the football team wins is $\frac{3}{14}$. What is the probability that both teams lose? $\frac{88}{210}$

8. The probability that Rafael arrives late at work on a given day is equal to 0.1.
What is the probability that Rafael

a) arrives on time on three consecutive days? ***0.729***

b) arrives late only once in three consecutive days? ***0.243***

Evaluation 5

1. Three candidates Angela, Brittney and Claudia apply for the position of secretary of a union.

Each voting member must write, in order, his preference for the choice of each candidate.

a) 1. How many different ways of voting are there? ***6 ways***

2. Enumerate the different ways.

(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A).

b) The results of the votes have been compiled.

We observe that 20 members chose the order (A, B, C), 12 members chose the order (C, A, B), 10 members chose the order (B, A, C) and 8 members chose the order (A, C, B).

Preference table

Number of members	*20*	*12*	*10*	*8*
1st choice	*A*	*C*	*B*	*A*
2nd choice	*B*	*A*	*A*	*C*
3rd choice	*C*	*B*	*C*	*B*

c) Determine the winner under

1. a majority ballot. ***A wins***
2. a plurality ballot. ***A wins***
3. Borda's method. ***A(78 pts); B(40 pts); C(32 pts); A wins***
4. Condorcet's criterion. ***A wins***
5. an elimination ballot. ***A wins***

2. Using the preference table on the right, determine the winner under

1. a majority ballot. ***Nobody wins***
2. a plurality ballot. ***A wins***
3. Borda's method. ***A(70 pts); B(42 pts);***
C(38 pts); A wins
4. Condorcet's criterion. ***A wins***
5. an elimination ballot. ***A wins***

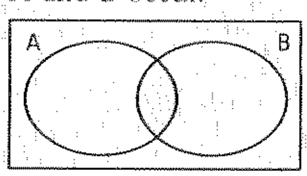
Preference table

Numbers of members	10	14	16	10
1st choice	A	C	B	A
2nd choice	B	A	A	C
3rd choice	C	B	C	B

3. Let A and B be two events associated with a random experiment.

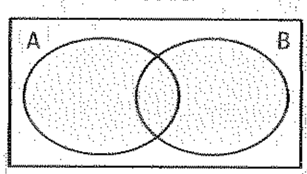
1. Represent the following events using a Venn diagram.
2. Describe each event using the operators ∩, ∪ or ‾.

a) A and B occur.



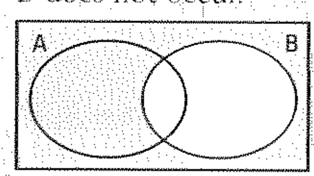
A ∩ B

b) A or B occurs.



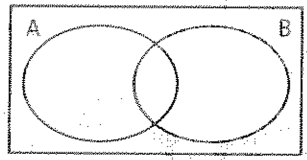
A ∪ B

c) B does not occur.



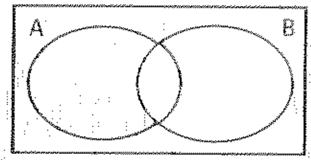
$\overline{B}$

d) A occurs and B does not occur.



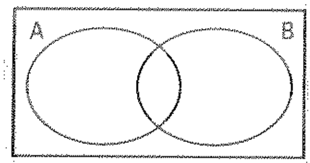
$A \cap \overline{B}$

e) A does not occur and B occurs.



$\overline{A} \cap B$

f) A and B do not occur.



$\overline{A \cup B}$

4. A statistical study showed that when a customer enters a convenience store, the probability that she will buy milk is 0.20, that she will buy beer is 0.30 and that she will buy both products is 0.05. A customer enters a convenience store. Calculate the probability that the customer will buy

a) at least one of these two products. *0.45*

b) only beer. *0.25*

c) milk if she buys beer. $\frac{1}{6}$

d) only one of these two products. *0.40*

5. According to the weather forecast, the probability of precipitations is 60 % for Saturday and 70 % for Sunday. Calculate the probability of the following events.

a) It rains both days. *0.42*

b) It rains on Saturday only. *0.18*

c) It rains only one of the two days. *0.46*

d) It doesn't rain during the weekend. *0.12*

6. At archery practice, Julian hits the target three out of five times. He shoots three times. Compute the probability of the following events.

a) He misses the target with each shot. $\frac{8}{125}$

b) He hits the target only once. $\frac{36}{125}$

c) He hits the target twice. $\frac{54}{125}$

d) He hits the target with every shot. $\frac{27}{125}$

7. Every morning, to get to work, an employee has a choice between two means of transportation: bus or car. After many years, he realizes that he takes the bus 4 out of 10 times and the car 6 out of 10 times. Moreover, he observed that when he takes the bus, he has a 2 out of 10 chance of arriving late for work, while he has only a one out of 10 chance of arriving late when he takes the car.

a) What is the probability that the employee arrives late at work? *0.14*

b) Today, this employee arrived late for work. What is the probability that he took his car?

$\frac{3}{7}$

SYMBOLS

$\mathbb{N}$	set of natural numbers
$\mathbb{N}^*$	set of non-negative natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Z}_+$	set of non-negative integers
$\mathbb{Z}_-$	set of non-positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Q}'$	set of irrational numbers
$\mathbb{R}$	set of real numbers
$\in$	belongs to
$\subseteq$	is a subset of
$\notin$	does not belong to
$\nsubseteq$	is not a subset of
$=$	is equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$<$	is less than
$>$	is greater than
$\leqslant$	is less than or equal to
$\geqslant$	is greater than or equal to
$\forall x$	for all x
$\Rightarrow$	logically implies
$\Leftrightarrow$	is logically equivalent
$[a, b]$	closed interval
$[a, b[$	left-closed and right-open interval
$]a, b]$	left-open and right-closed interval
$]a, b[$	open interval
$]-\infty, a]$	left-unbounded and right-closed interval
$]-\infty, a[$	left-unbounded and right-open interval
$[a, +\infty[$	left-closed and right-unbounded interval
$]a, +\infty[$	left-open and right-unbounded interval
$\varnothing$	empty set, impossible event
AB	line AB
$\overline{\text{AB}}$	line segment AB
$\text{m}\overline{\text{AB}}$	measure of line segment AB
$\cong$	is congruent to
$\angle$AOB	angle AOB
m $\angle$AOB	measure of angle AOB
$\perp$	is perpendicular to
//	is parallel to

$\triangle ABC$	triangle ABC
$M(x, y)$	coordinates of point M
$\sim$	is similar to
A_l	lateral area of a solid
A_b	area of the base of a solid
A_t	total area of a solid
V	volume of a solid
R^{-1}	inverse of relation R
$f(x)$	image of x under function f
dom f	domain of function f
ima f	image of function f
max f	maximum of function f
min f	minimum of function f
Ω	universal set, certain event
$A \cap B$	intersection of events A and B
$A \cup B$	union of events A and B
$\overline{A}$	event contrary to A
A'	event contrary to A
$A \backslash B$	event A minus event B
$P(A)$	probability of event A
$P(A \mid B)$	probability of event A given event B
$n!$	n factorial
$\{a, b\}$	edge ab or ba
$d(A)$	degree of vertex A
$d(A, B)$	distance between two vertices A and B
t:	translation
$r\,(0, \alpha)$	counterclockwise rotation of angle α centred at 0
$r\,(0, -\alpha)$	clockwise rotation of angle α centred at 0
s:	reflection
s_x:	reflection about the x-axis
s_y:	reflection about the y-axis
$s_{\boxslash}$:	reflection about the first bisector
$s_{\boxbslash}$:	reflection about the second bisector
$e_v\,(k)$:	vertical scaling of factor k
$e_h\,(k)$:	horizontal scaling of factor k
$h(0, k)$:	homothety centred at 0 with ratio k

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